Graph Theory

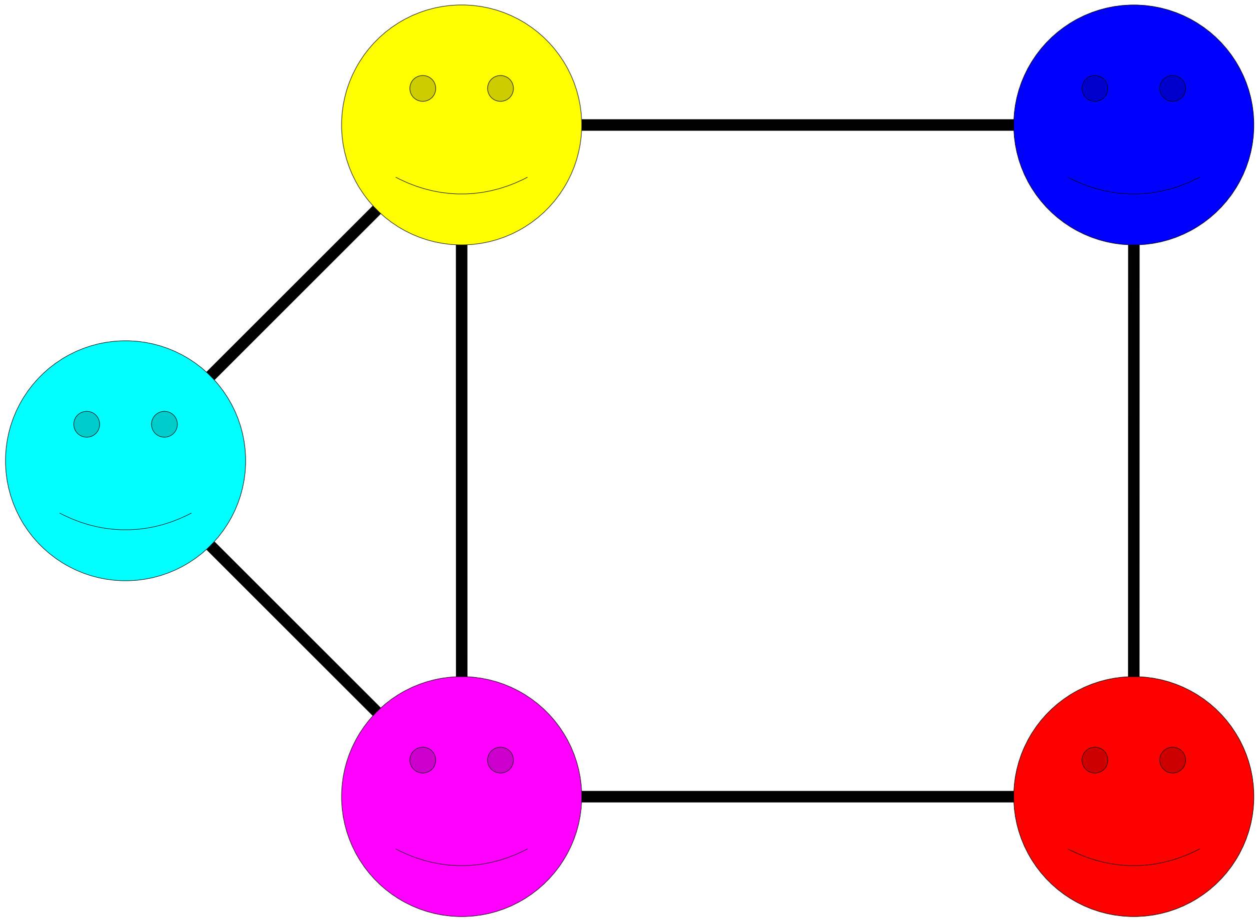
Part Two

Outline for Today

- ***Graph Complements***
  - Flipping what's in a graph.
- ***The Pigeonhole Principle***
  - A simple yet surprisingly effective fact.
- ***Graph Theory Party Tricks***
  - Cool tricks to try at your next group meeting.
- ***A Little Movie Puzzle***
  - Who watched what?

Recap from Last Time

A ***graph*** is a mathematical structure for representing relationships.



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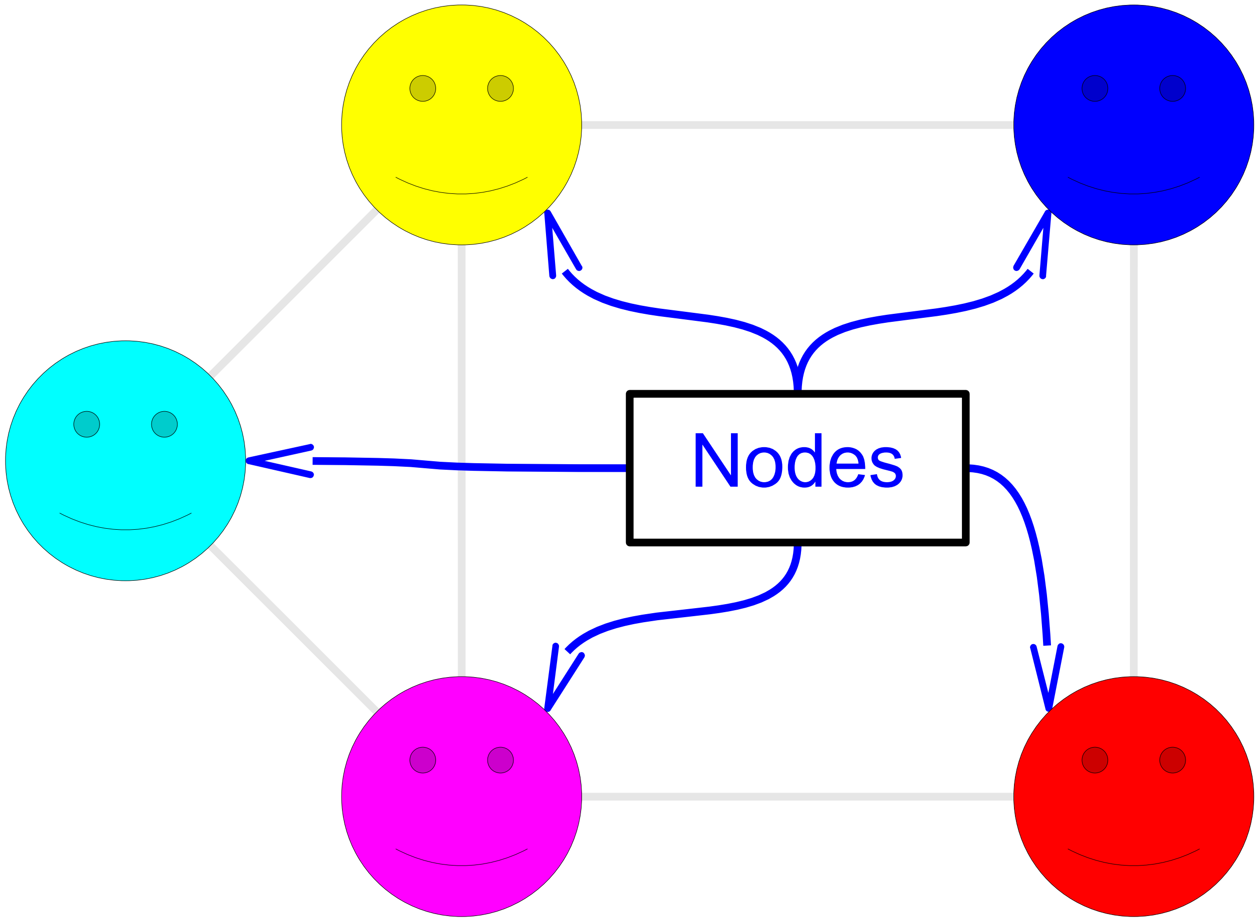

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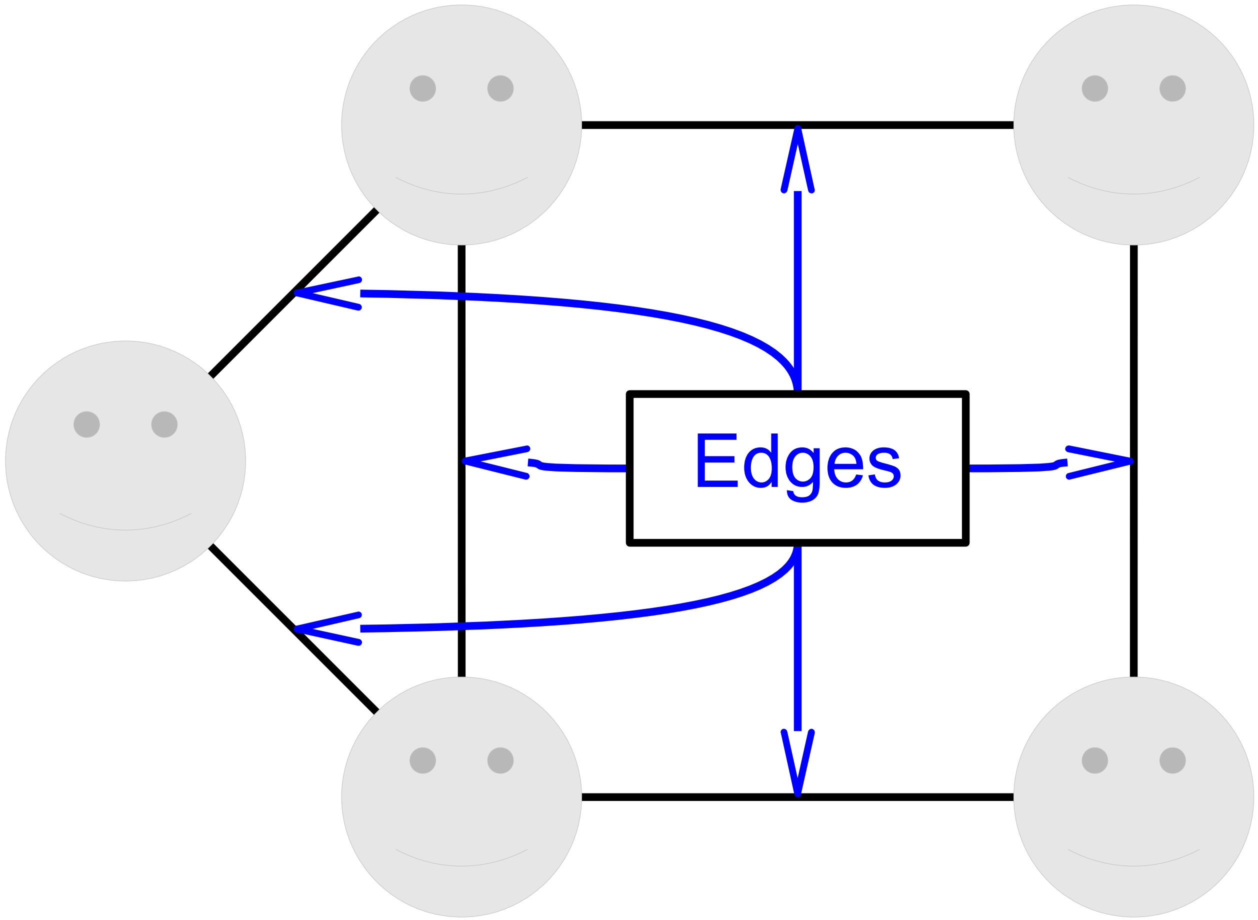

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Formalizing Graphs

- An ***unordered pair*** is a set $\{a, b\}$ of two elements $a \neq b$. (Remember that sets are unordered.)
  - For example, $\{0, 1\} = \{1, 0\}$
- An ***undirected graph*** is an ordered pair $G = (V, E)$, where
  - V is a set of nodes, which can be anything, and
  - E is a set of edges, which are *unordered* pairs of nodes drawn from V.
- A ***directed graph*** (or ***digraph***) is an ordered pair $G = (V, E)$, where
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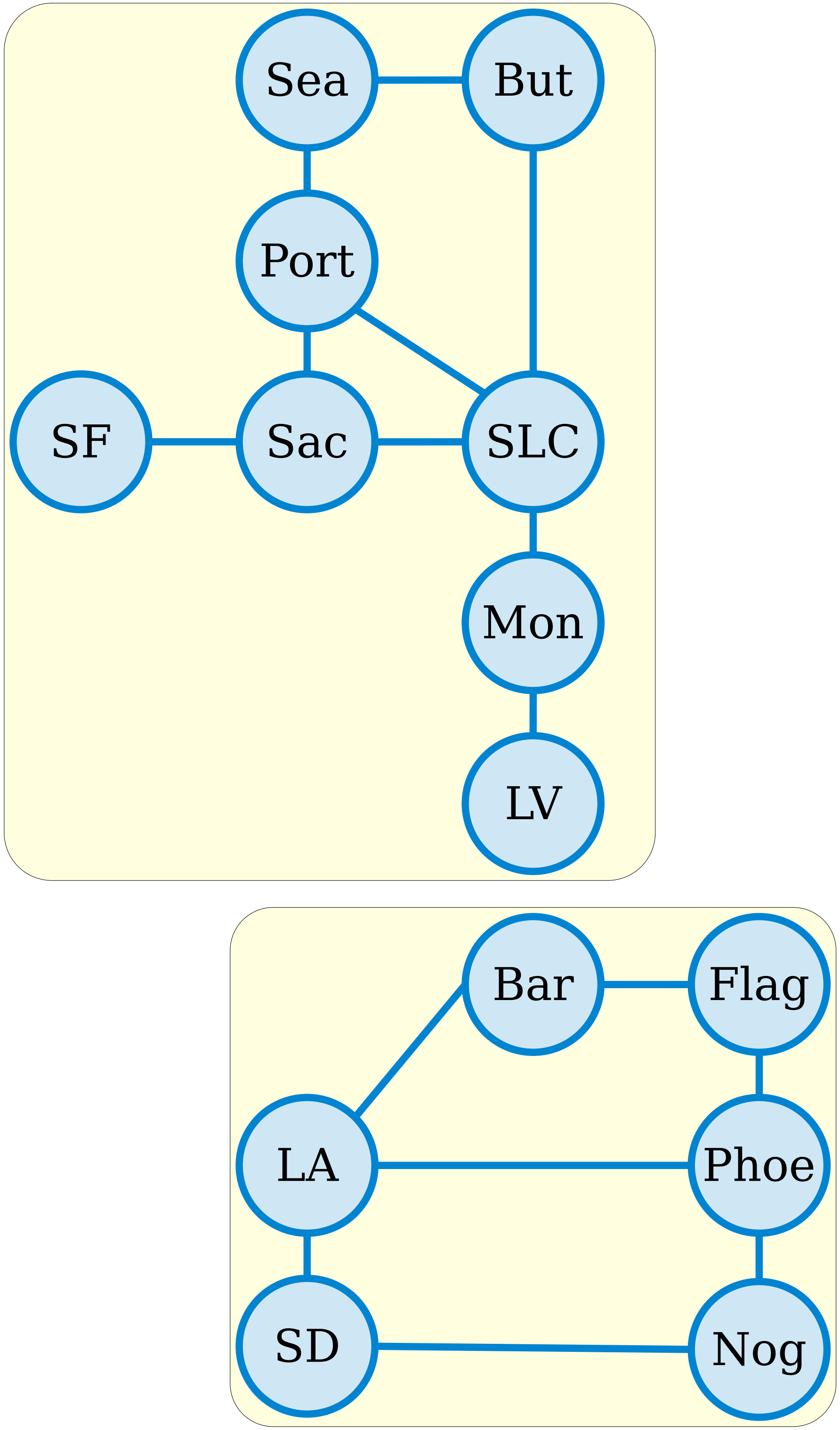
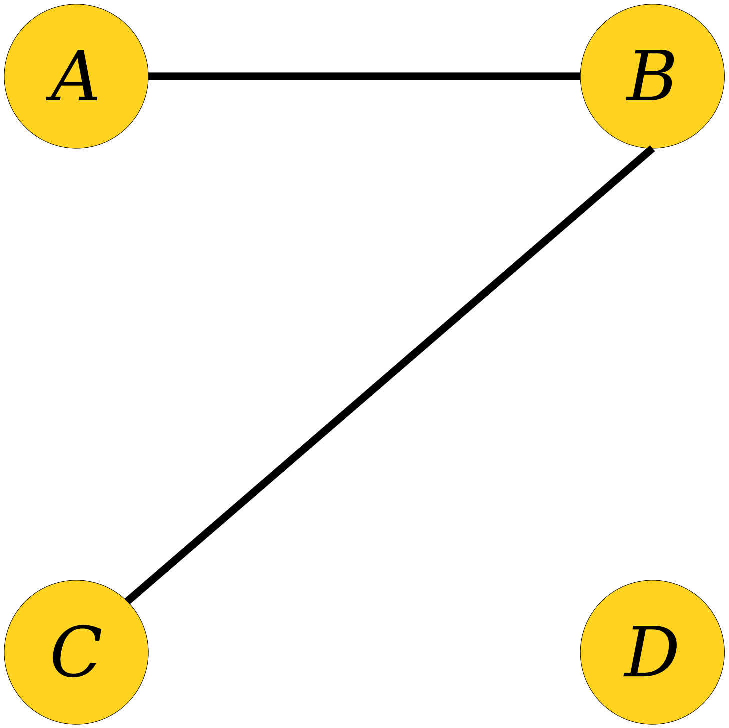


A graph G is called ***connected*** if all pairs of distinct nodes in G are reachable.

A ***connected component*** (or ***CC***) of G is a maximal set of mutually reachable nodes.

New Stuff!

Graph Complements

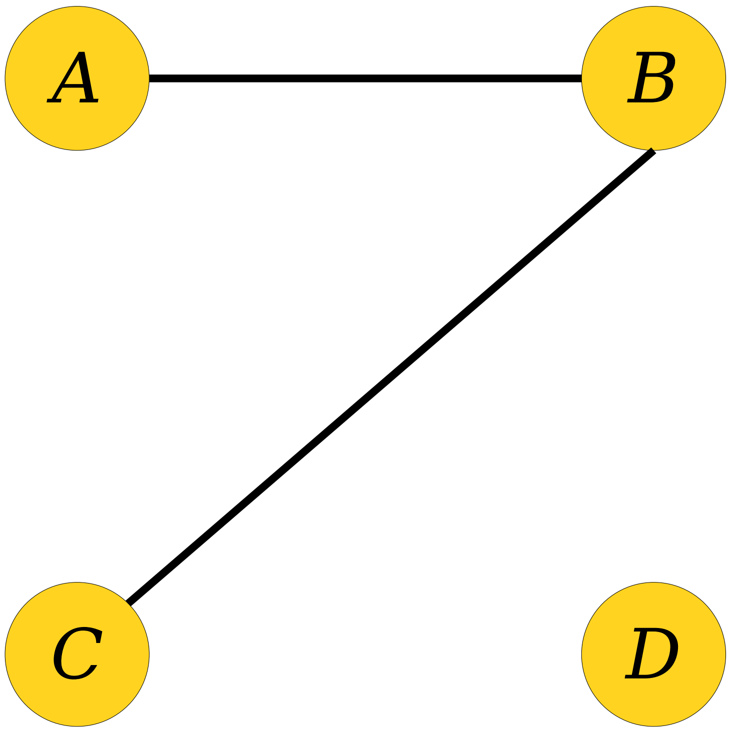


$G = (V, E)$
$V = \{ A, B, C, D \}$
$E = \{\{A, B\}, \{B, C\}\}$

Based on the definition below, what is G^c for this graph? Give your answer as sets V and E^c.

Respond at pollev.com/zhenglian740

Let $G = (V, E)$ be an undirected graph.
The ***complement of G*** is the graph $G^c = (V, E^c)$, where
$E^c = \{ \{u, v\} \mid u \in V, v \in V, u \neq v, \text{and } \{u, v\} \notin E \}$



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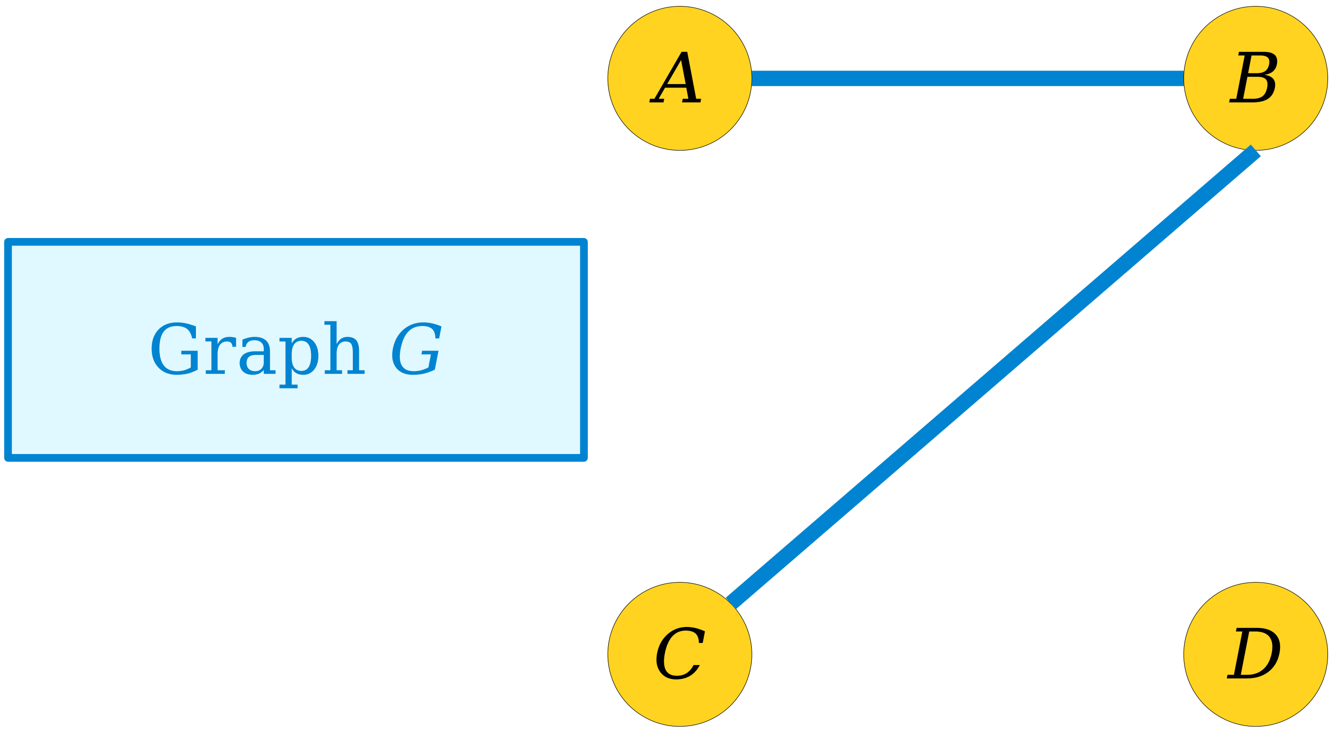

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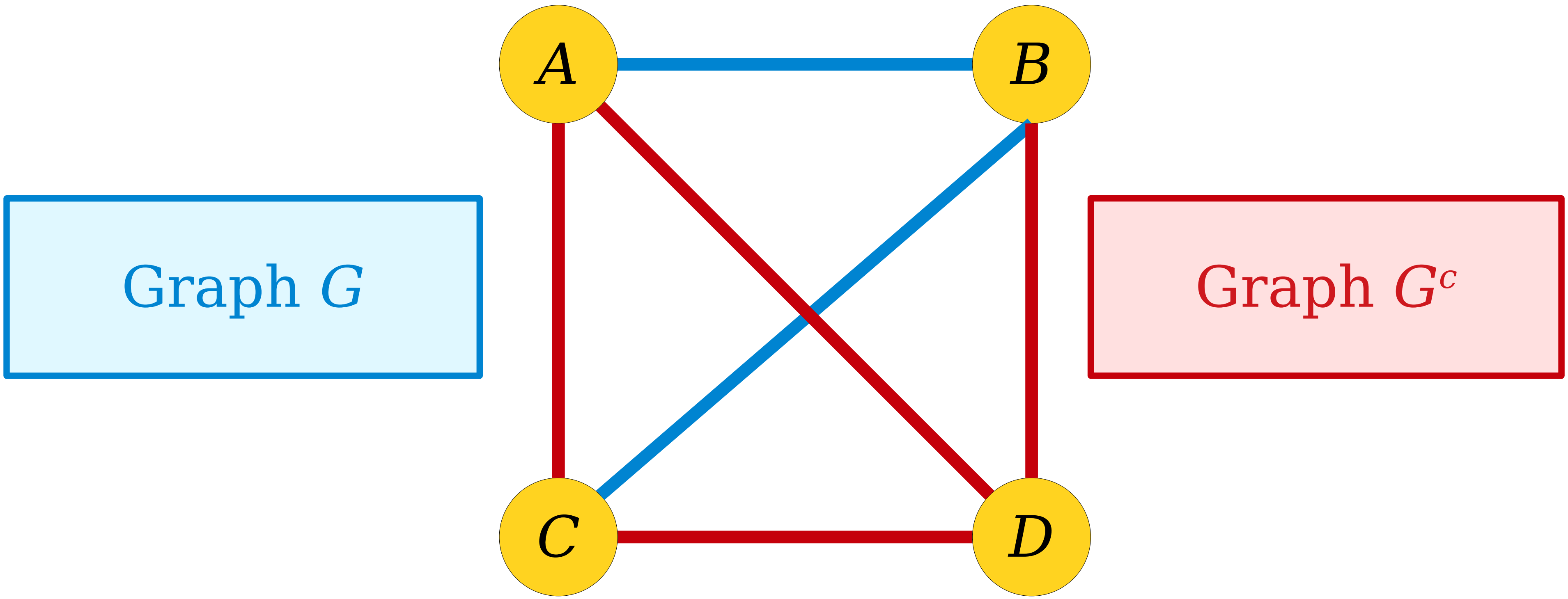

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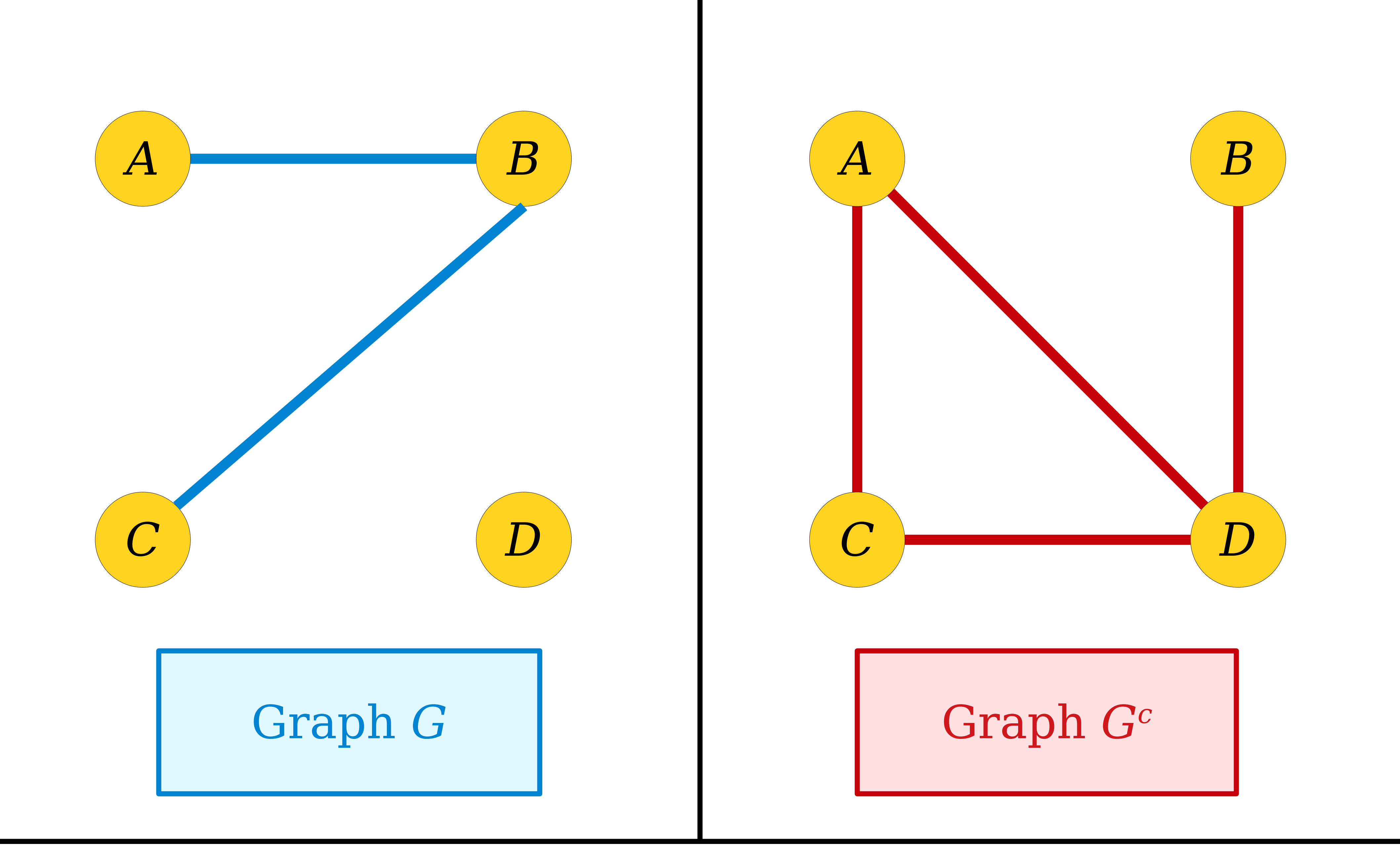

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Theorem: For any graph $G = (V, E)$, at least one of G and G^c is connected.

Proving a Disjunction

- We need to prove the statement

  G is connected $\lor$ G^c is connected.

- Here's a neat observation.
  - If G is connected, we're done.
  - Otherwise, G isn't connected, and we have to prove that G^c is connected.
- We will therefore prove

  G is not connected $\to$ G^c is connected.

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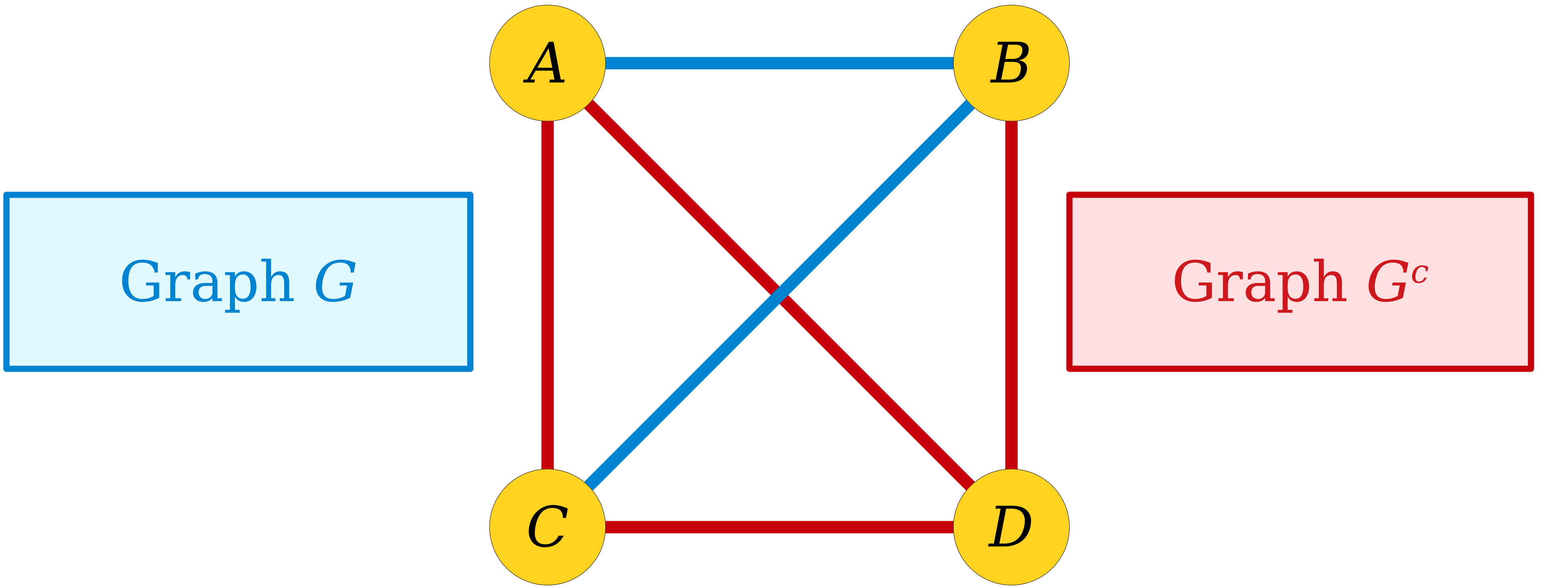

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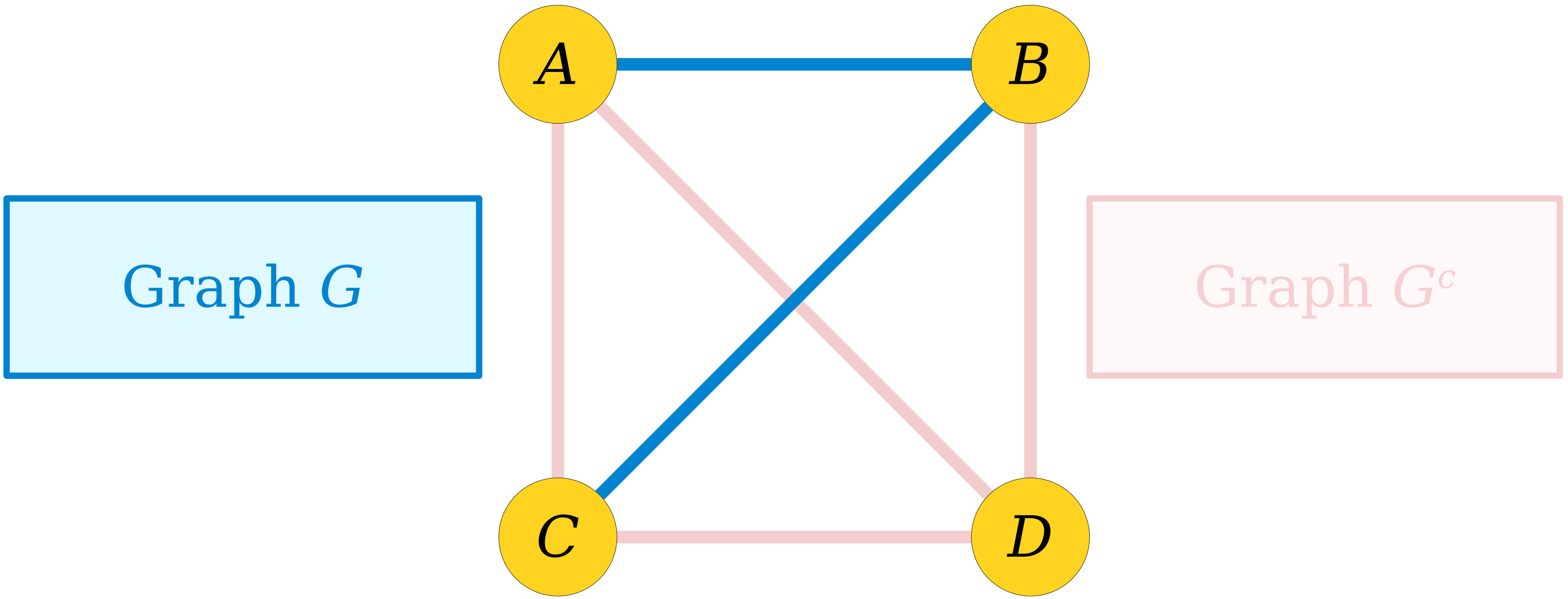

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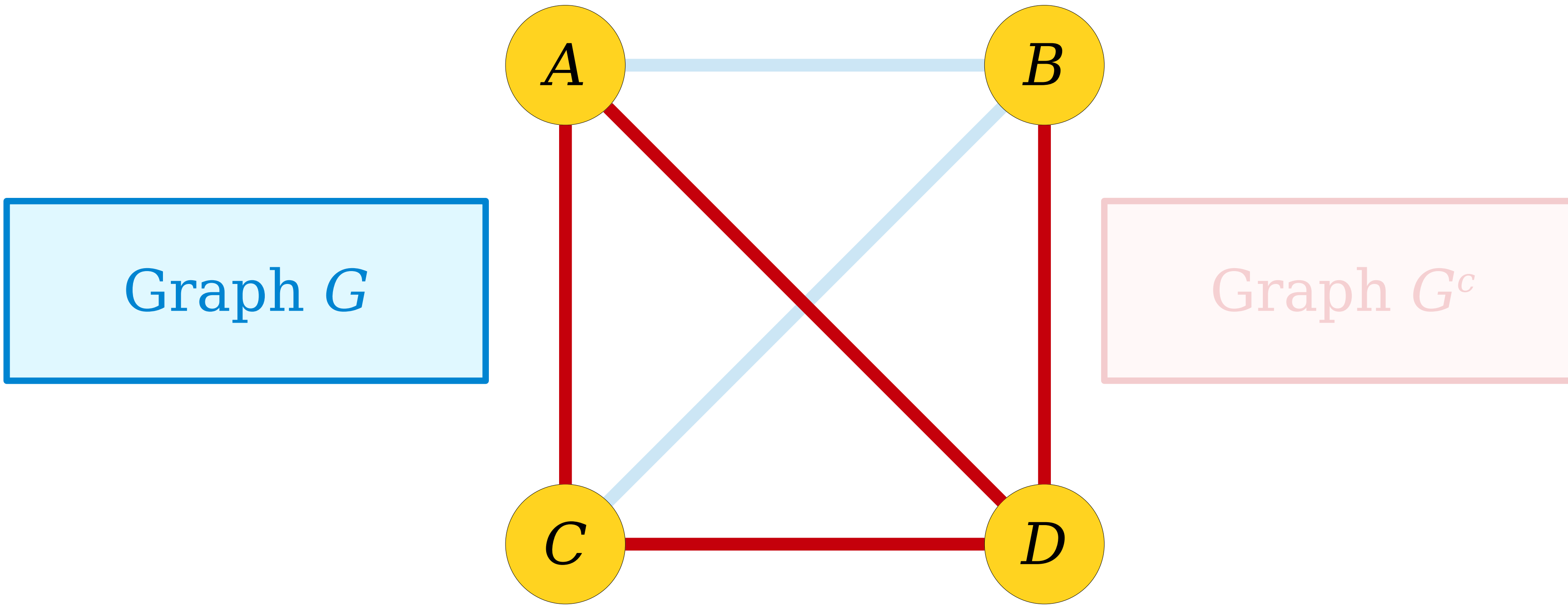

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Then you can reach all the nodes via red edges.

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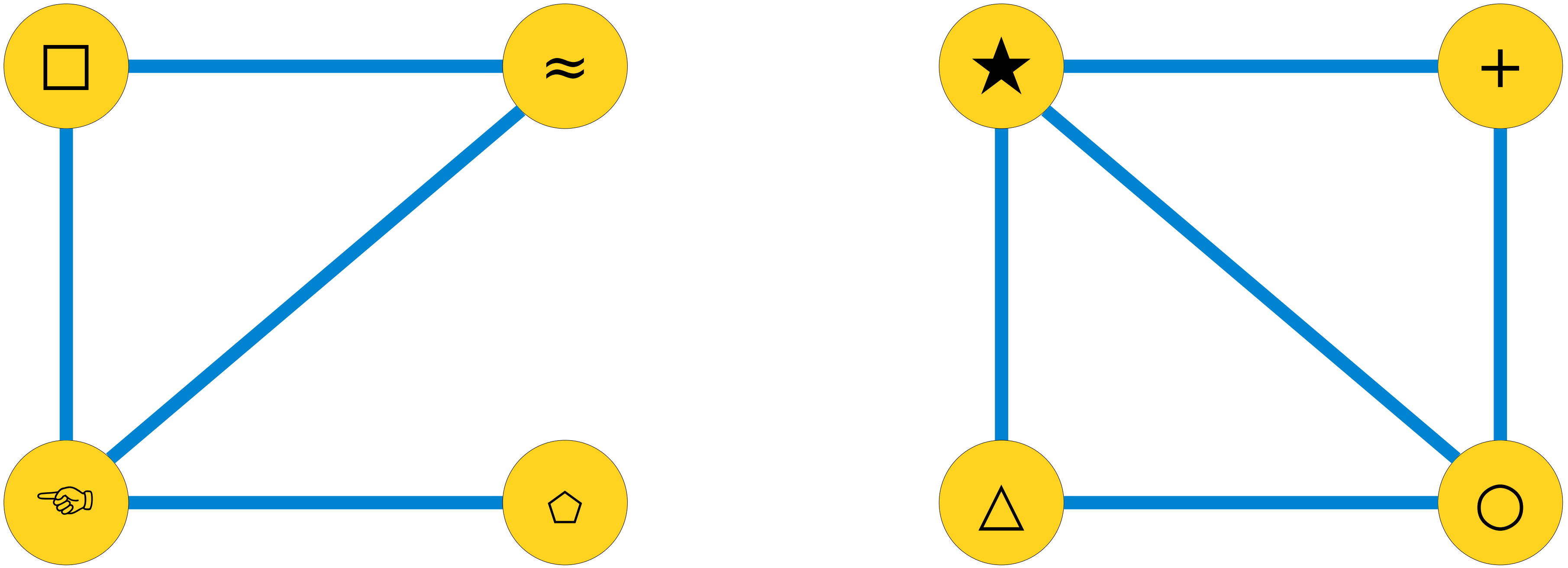

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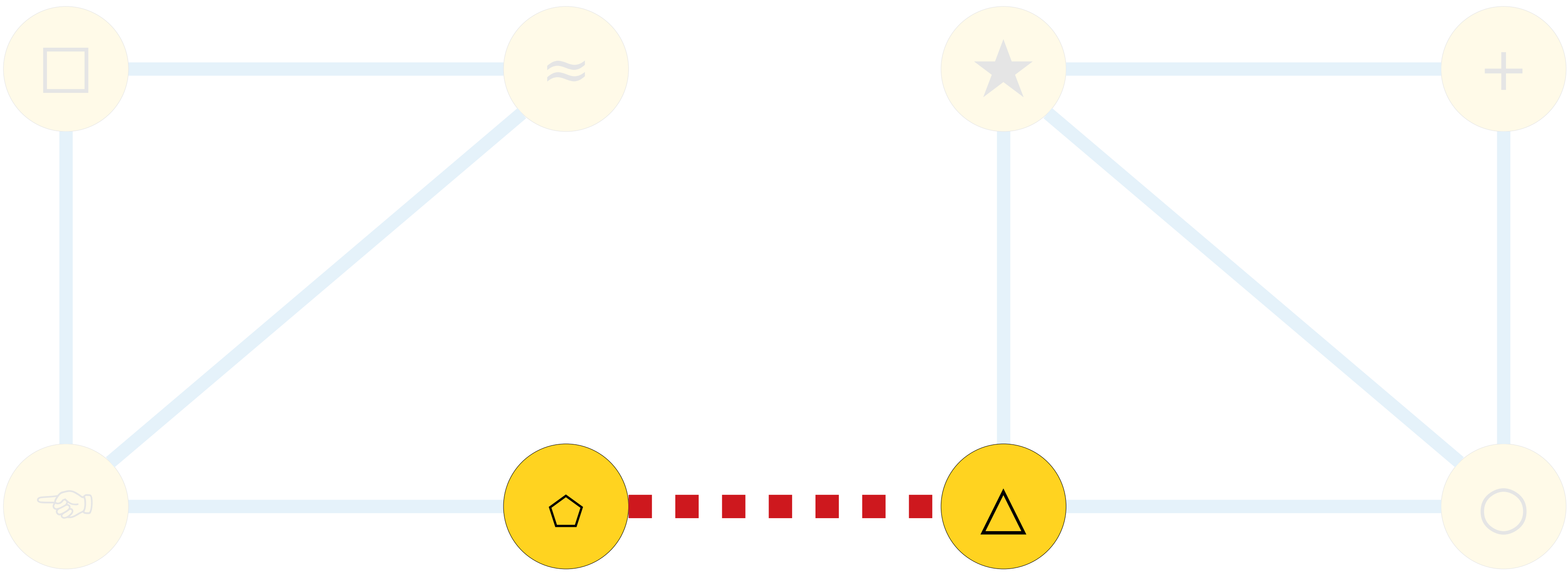
What happens if we look at two nodes that are *not* connected in G?



Observation: two nodes in G in different CC's of G become adjacent in G^c.

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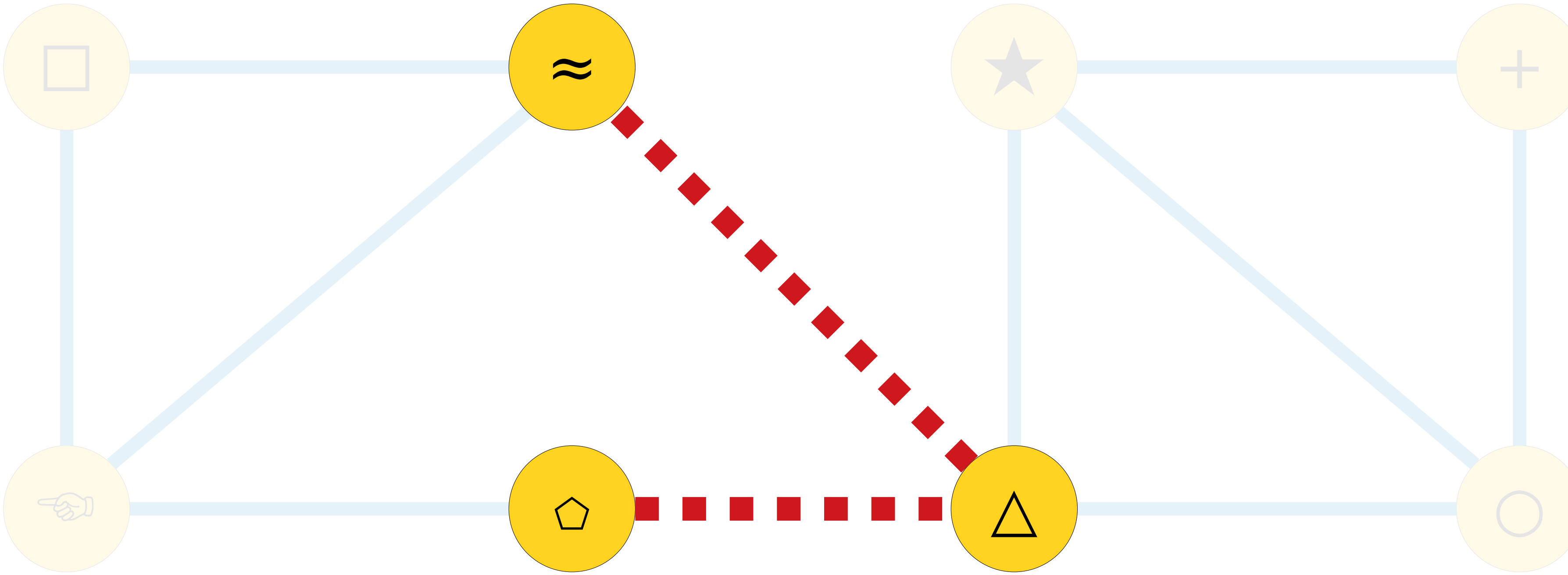
What happens if we look at two nodes that are connected in *the original graph*?



Observation: Any two nodes in G in the same CC can be "bridged" in G^c through a node in a different CC of G.

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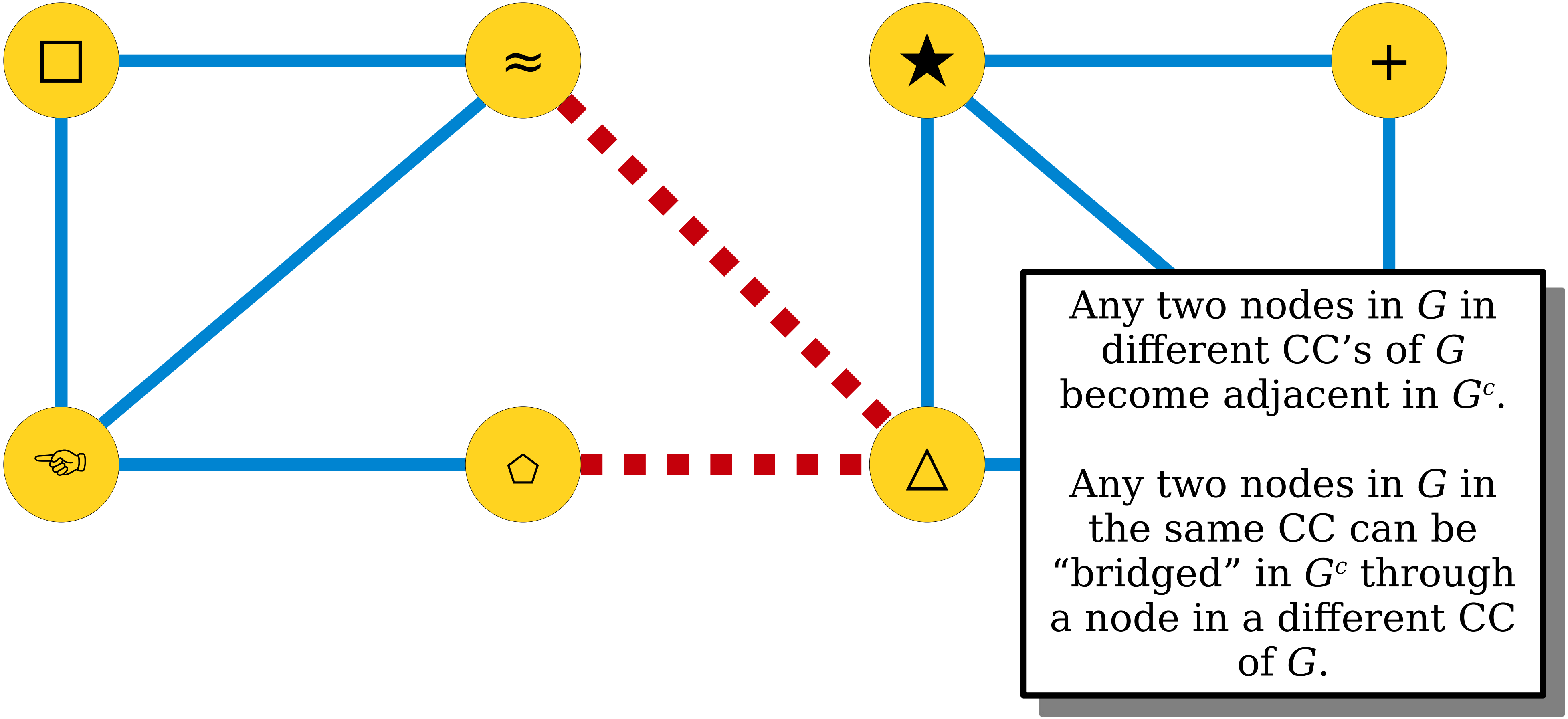

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The Pigeonhole Principle

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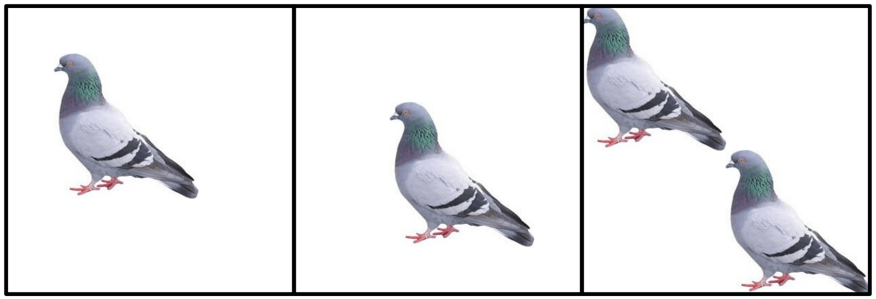

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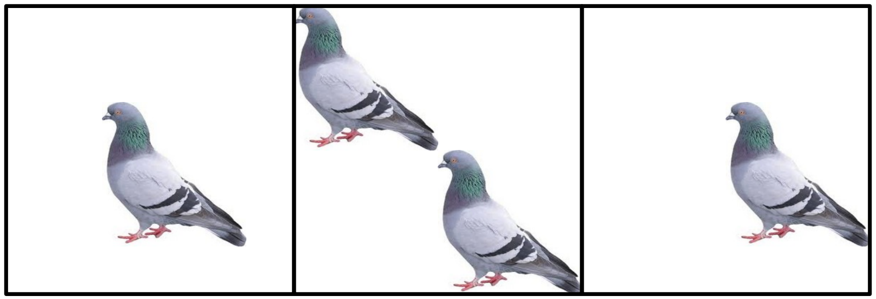

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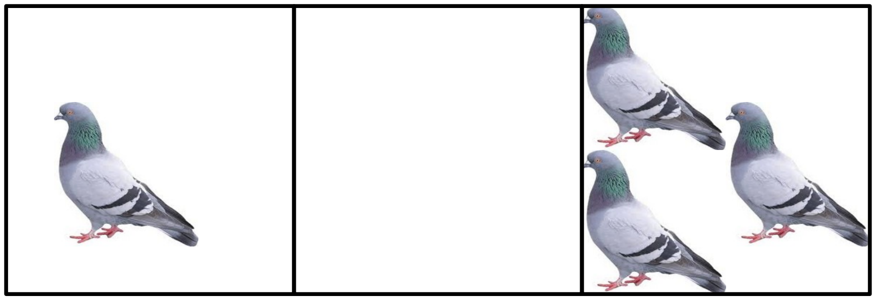

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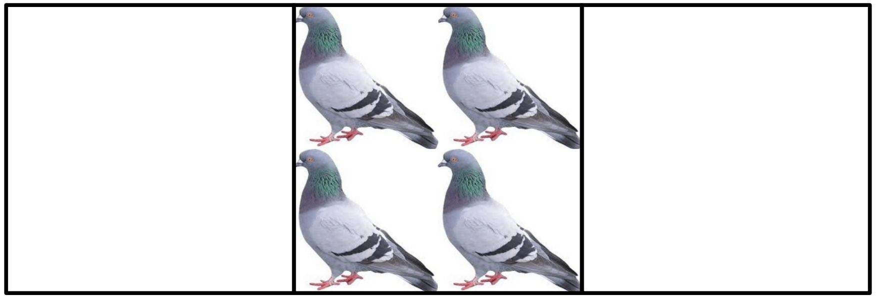
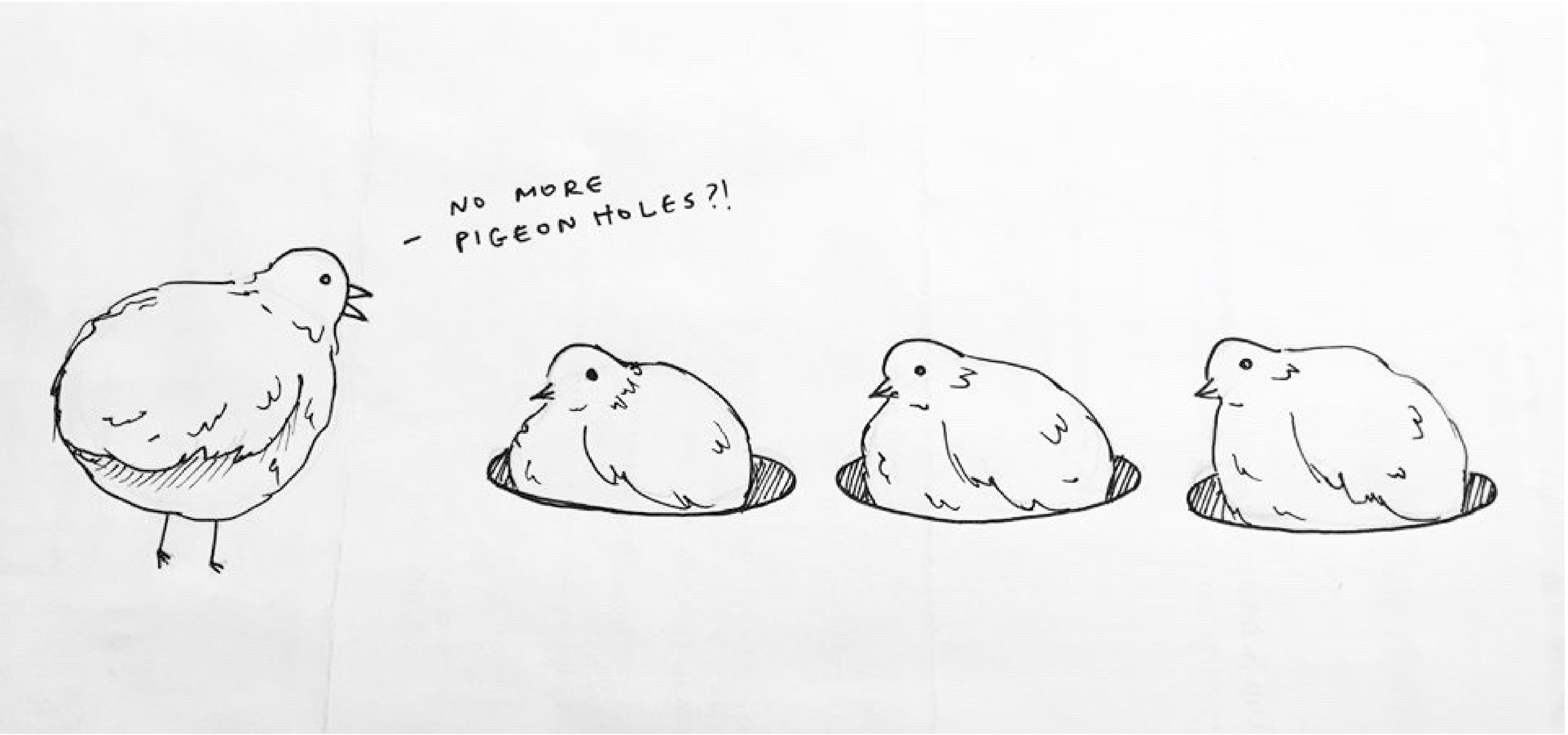




$m = 4, n = 3$

Some Simple Applications

- Any group of 367 people must have a pair of people that share a birthday.
  - 366 possible birthdays (pigeonholes).
  - 367 people (pigeons).
- Two people in San Francisco have the exact same number of hairs on their head.
  - Maximum number of hairs ever found on a human head is no greater than 500,000.
  - There are over 800,000 people in San Francisco.

Proving the Pigeonhole Principle

Theorem: If m objects are distributed into n bins and $m > n$, then there must be some bin that contains at least two objects.

Proof: Suppose for the sake of contradiction that, for some m and n where $m > n$, there is a way to distribute m objects into n bins such that each bin contains at most one object.

Number the bins 1, 2, 3, …, n and let x_i denote the number of objects in bin i. There are m objects in total, so we know that

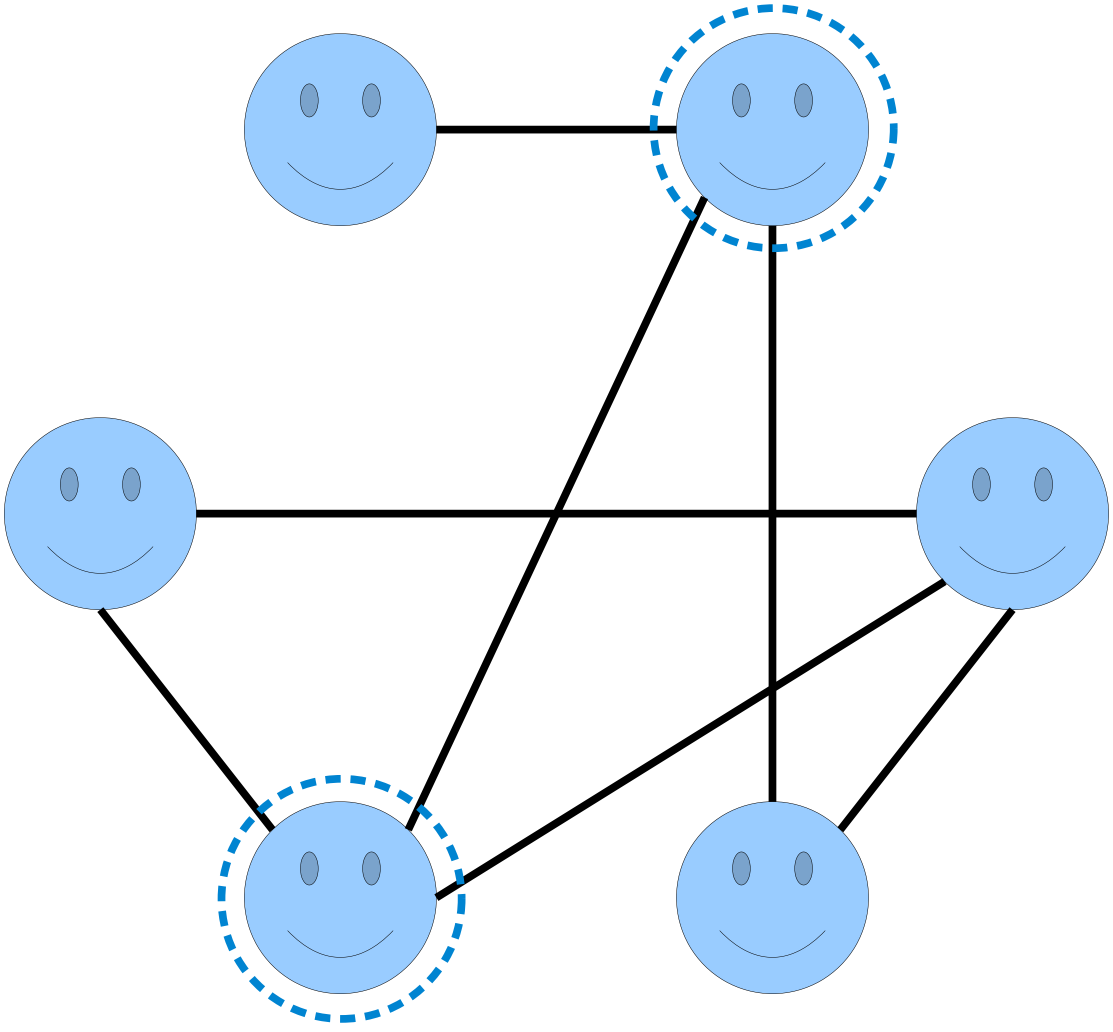
$$m = x_1 + x_2 + \ldots + x_n.$$

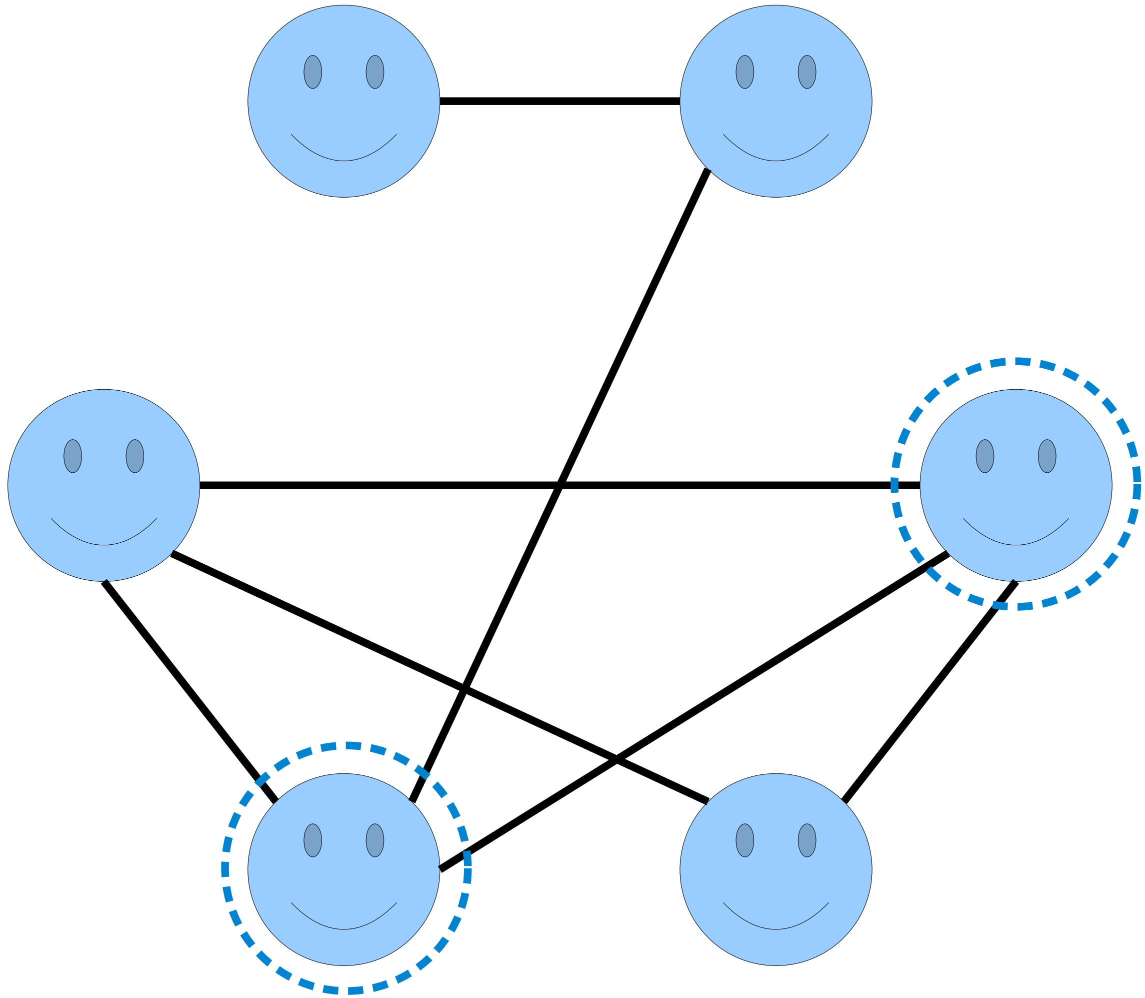
Since each bin has at most one object in it, we know $x_i \leq 1$ for each i. This means that

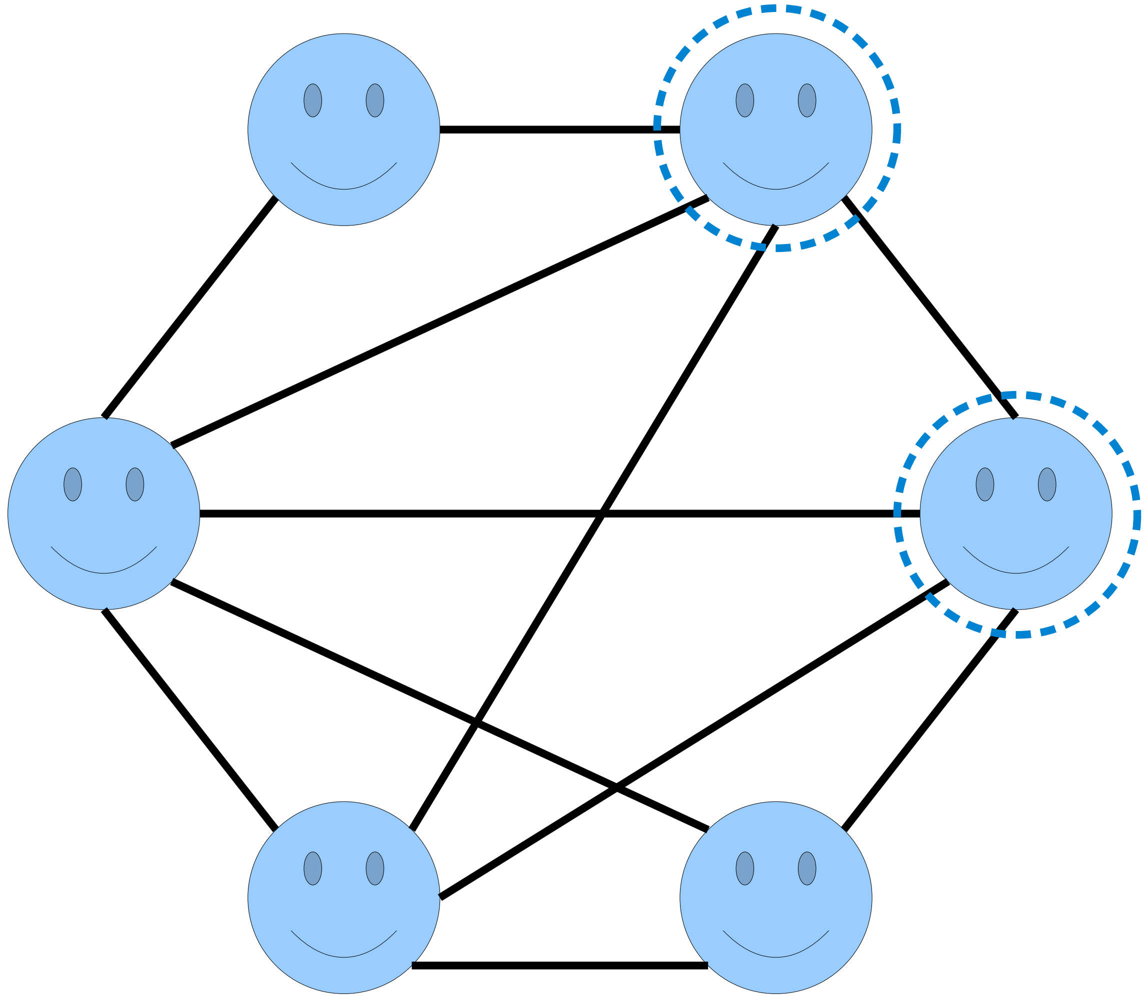
$$\begin{aligned} m &= x_1 + x_2 + \ldots + x_n \\ &\leq 1 + 1 + \ldots + 1 \quad \textit{(n times)} \\ &= n. \end{aligned}$$

This means that $m \leq n$, contradicting that $m > n$. We've reached a contradiction, so our assumption must have been wrong. Therefore, if m objects are distributed into n bins with $m > n$, some bin must contain at least two objects. ■

Pigeonhole Principle Party Tricks

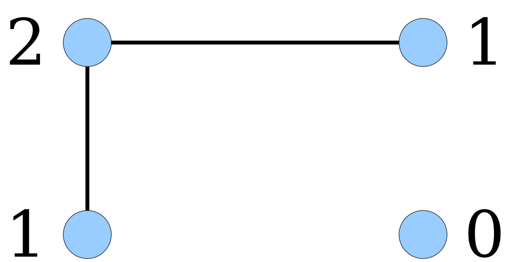


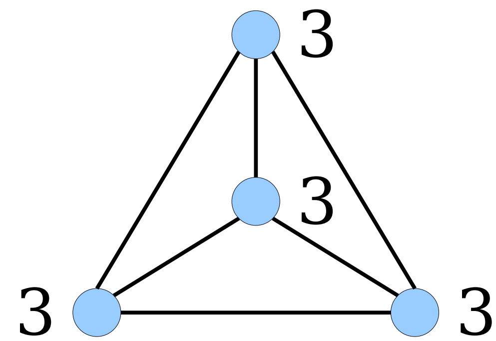




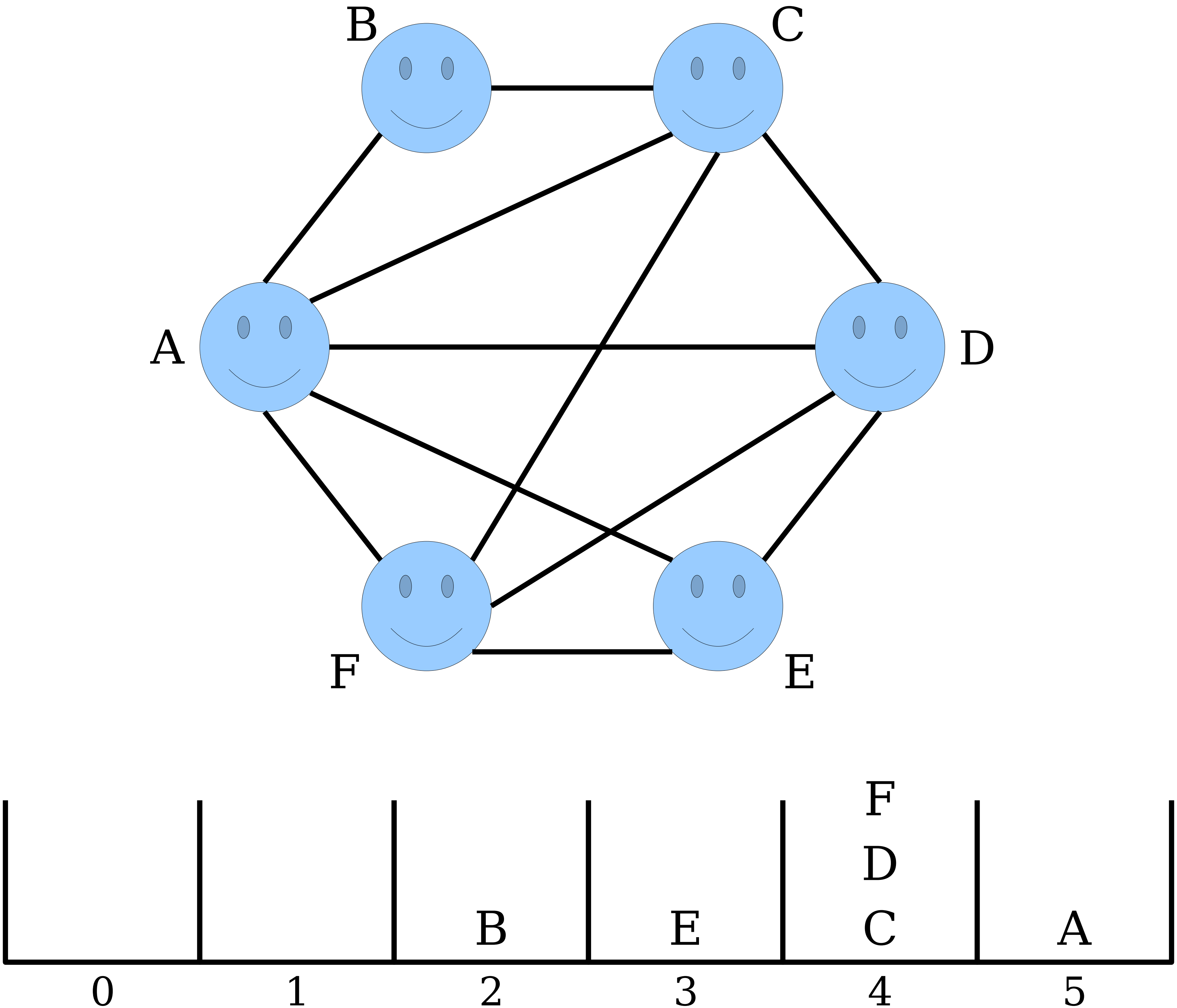
Degrees

- The ***degree*** of a node *v* in a graph is the number of nodes that *v* is adjacent to.





- ***Theorem:*** Every graph with at least two nodes has at least two nodes with the same degree.
  - Equivalently: at any party with at least two people, there are at least two people with the same number of friends at the party.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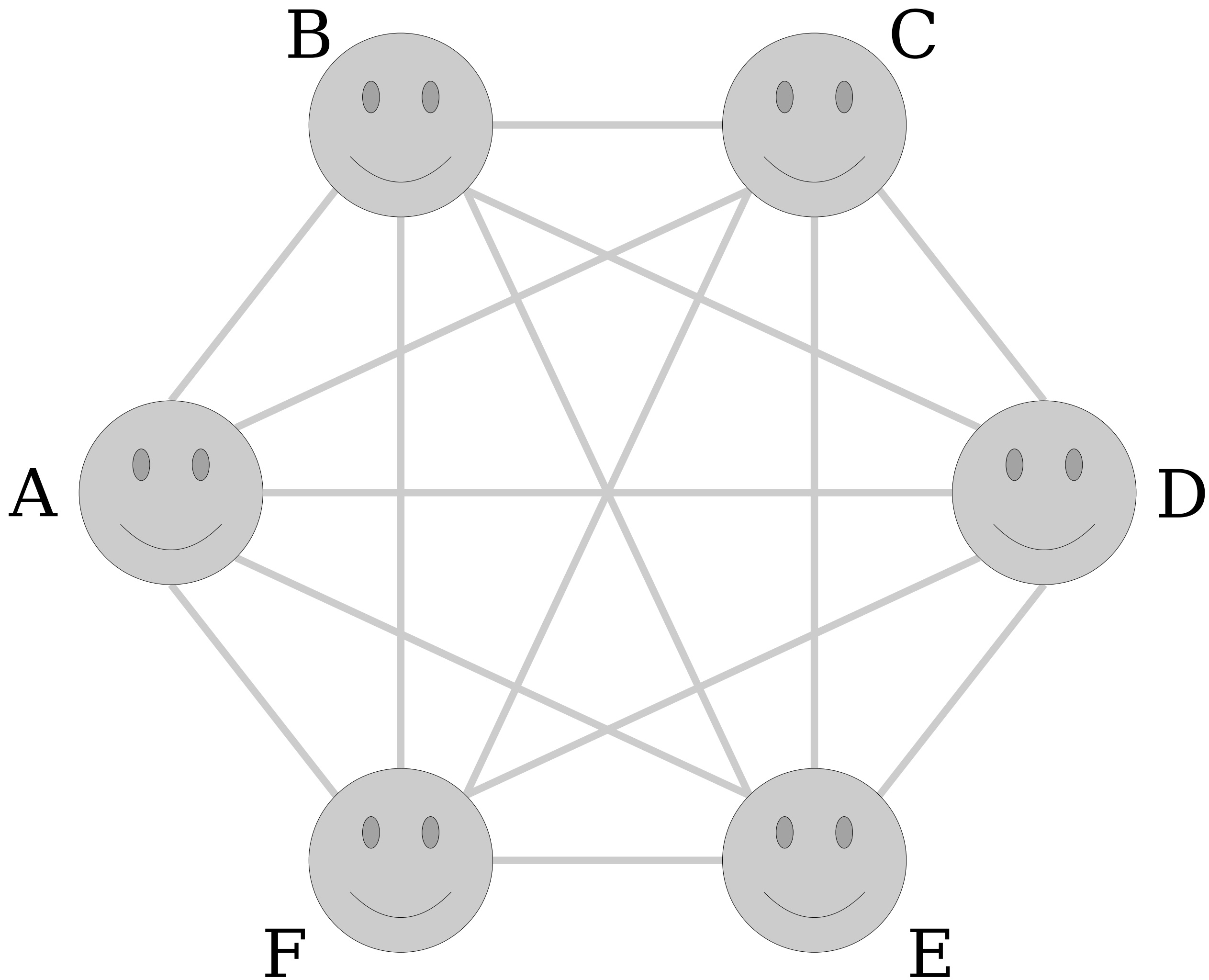
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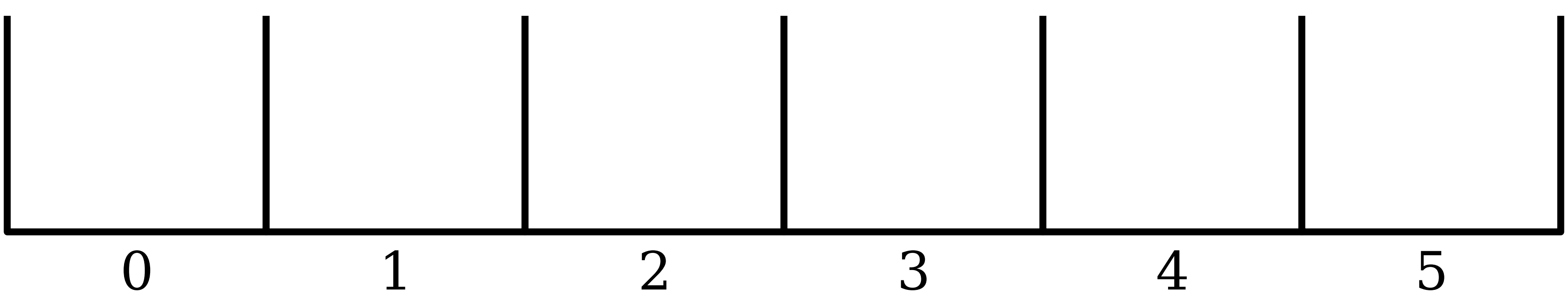
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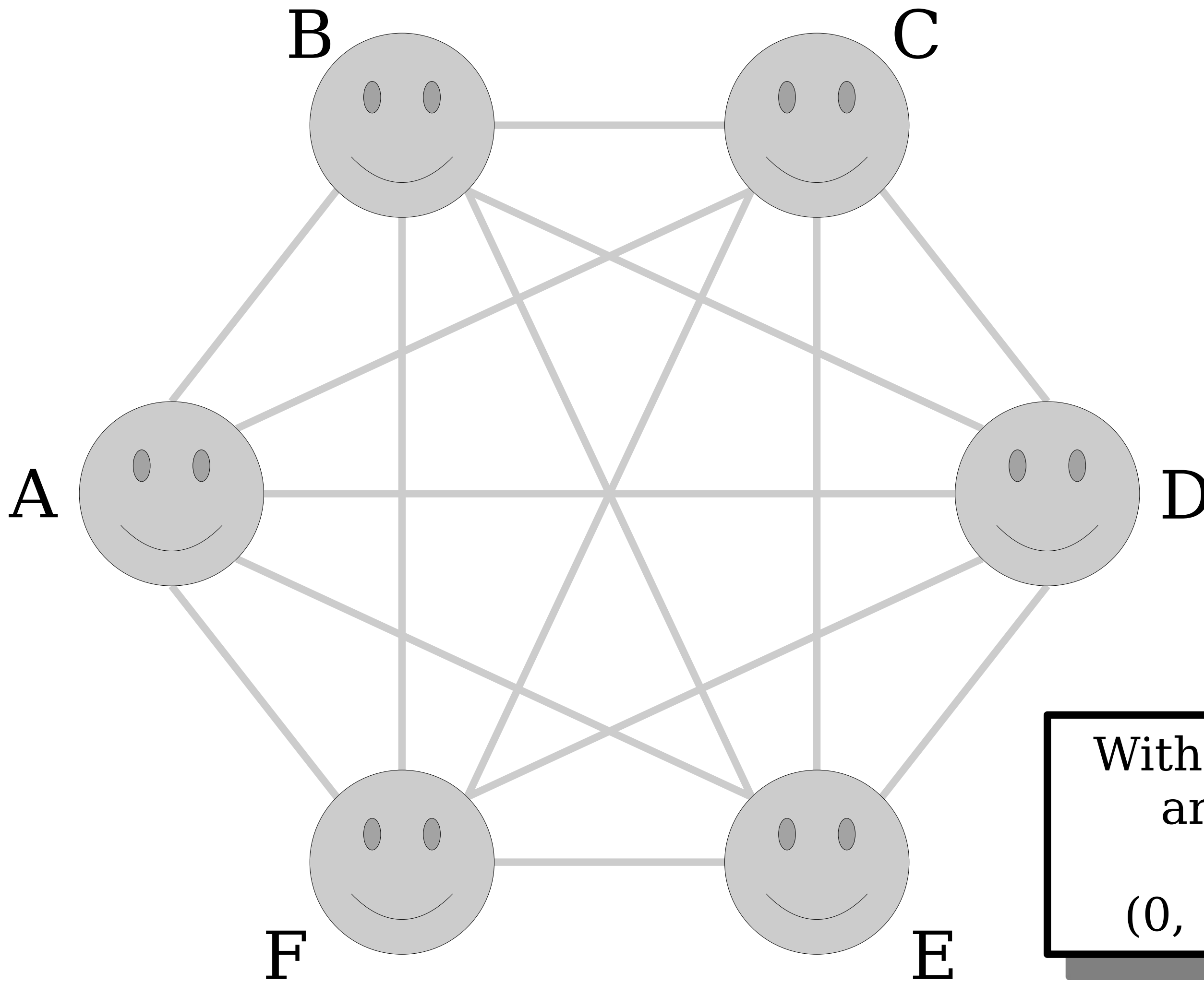
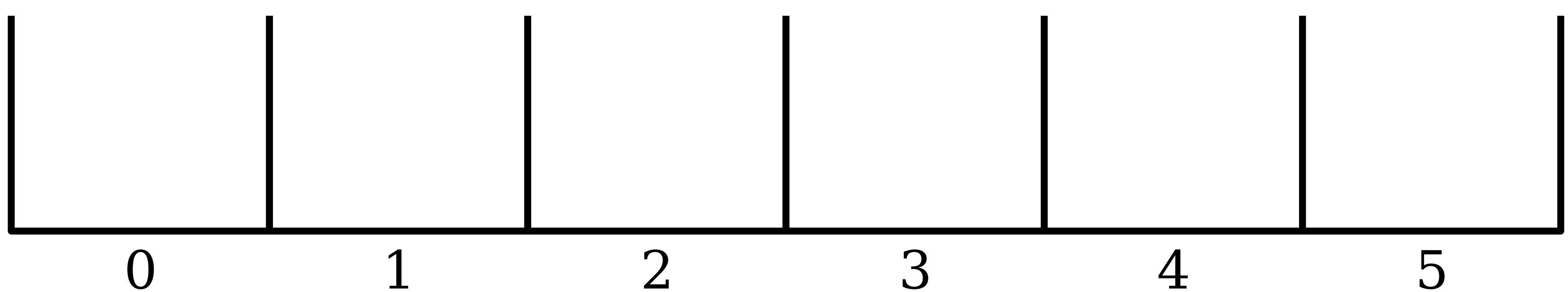
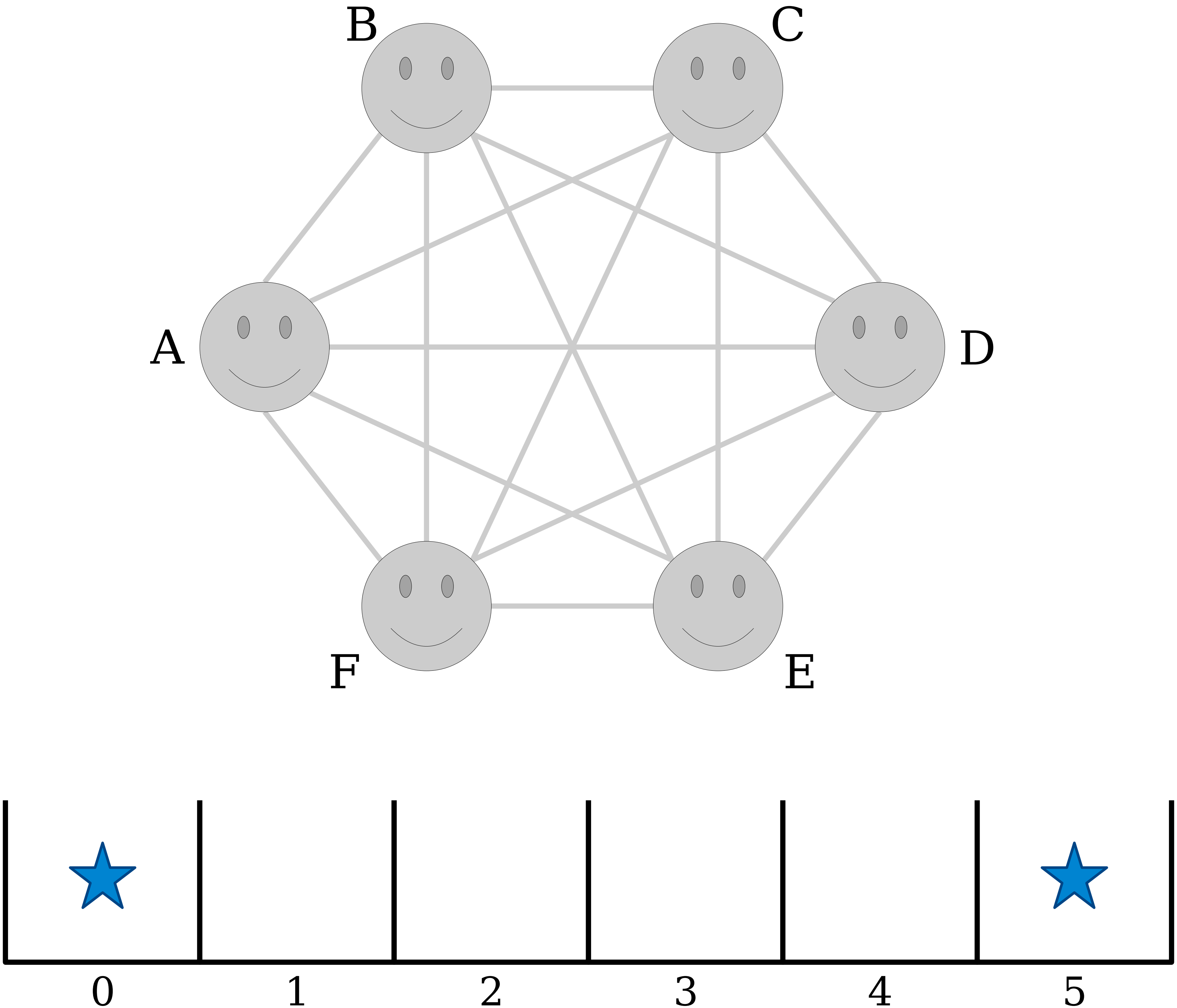


With n nodes, there are n possible degrees ($0, 1, 2, \ldots, n - 1$)




A
B
C
D
E
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0
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2
3
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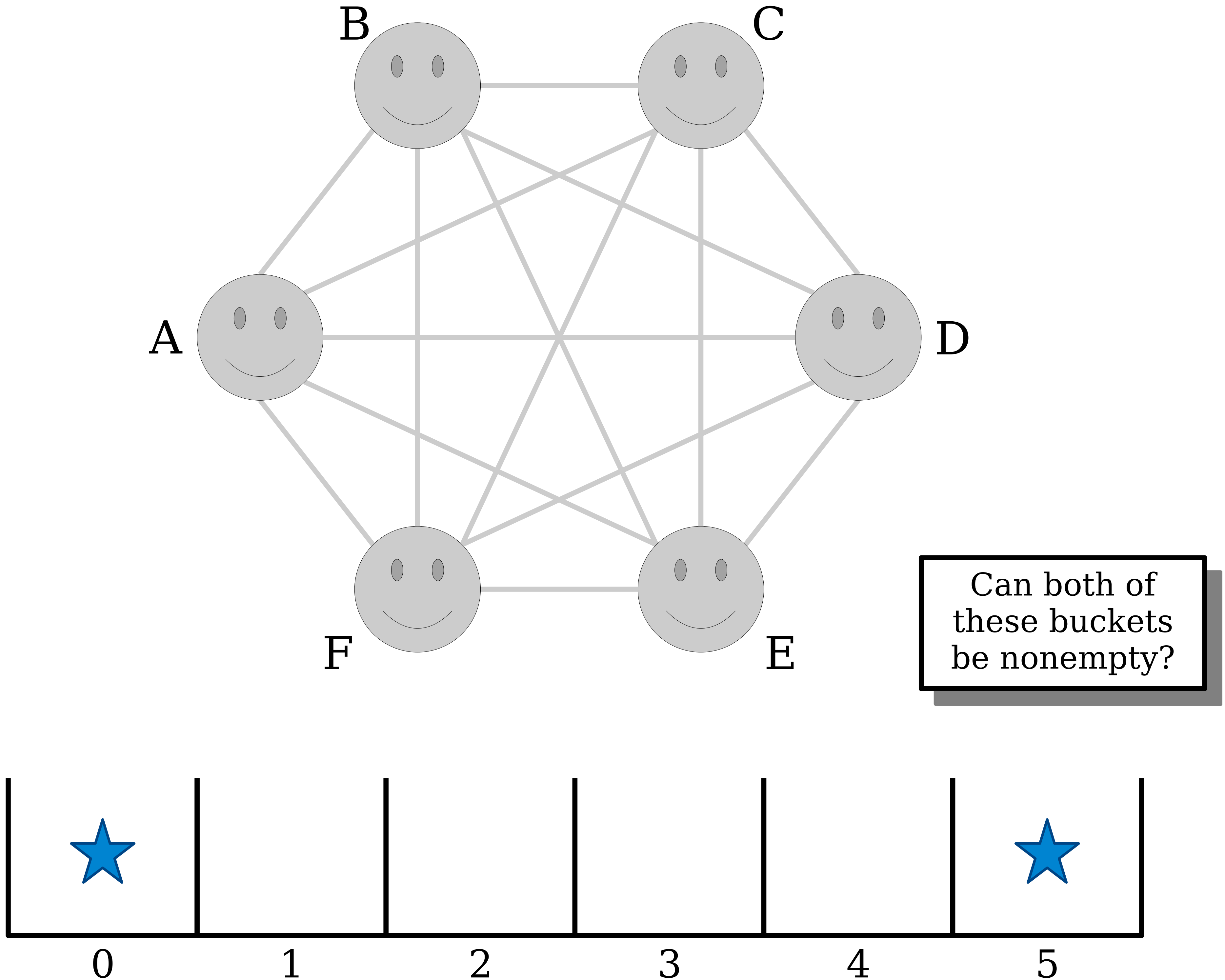
5


B
C
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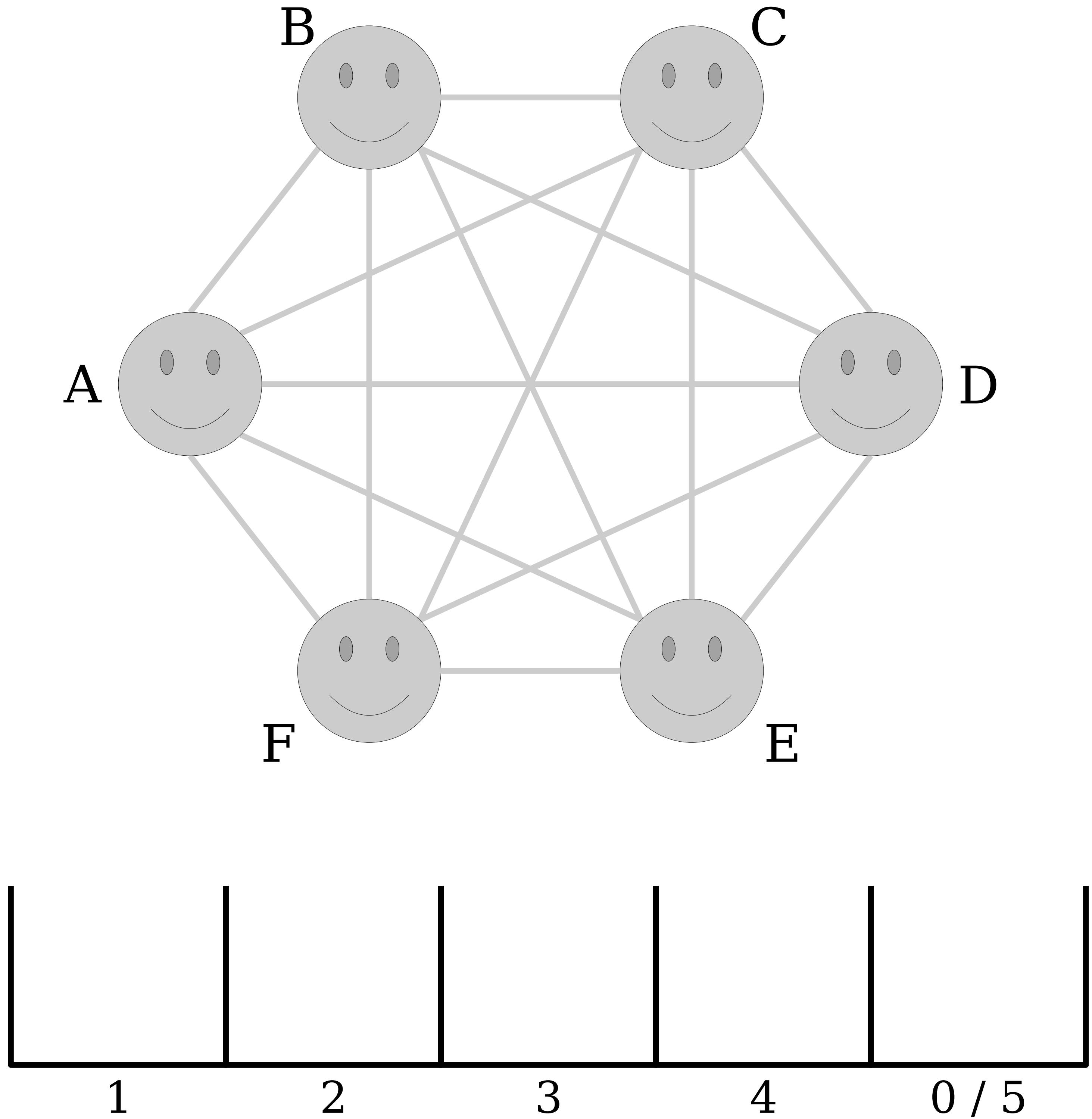
E
Can both of these buckets be nonempty?
0
1
2
3
4
5


B
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1
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0 / 5

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We therefore see that the possible options for degrees of nodes in G are either drawn from $0, 1, \ldots, n - 2$ or from $1, 2, \ldots, n - 1$.

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We therefore see that the possible options for degrees of nodes in G are either drawn from 0, 1, …, $n - 2$ or from 1, 2, …, $n - 1$. In either case, there are n nodes and $n - 1$ possible degrees, so by the pigeonhole principle two nodes in G must have the same degree.

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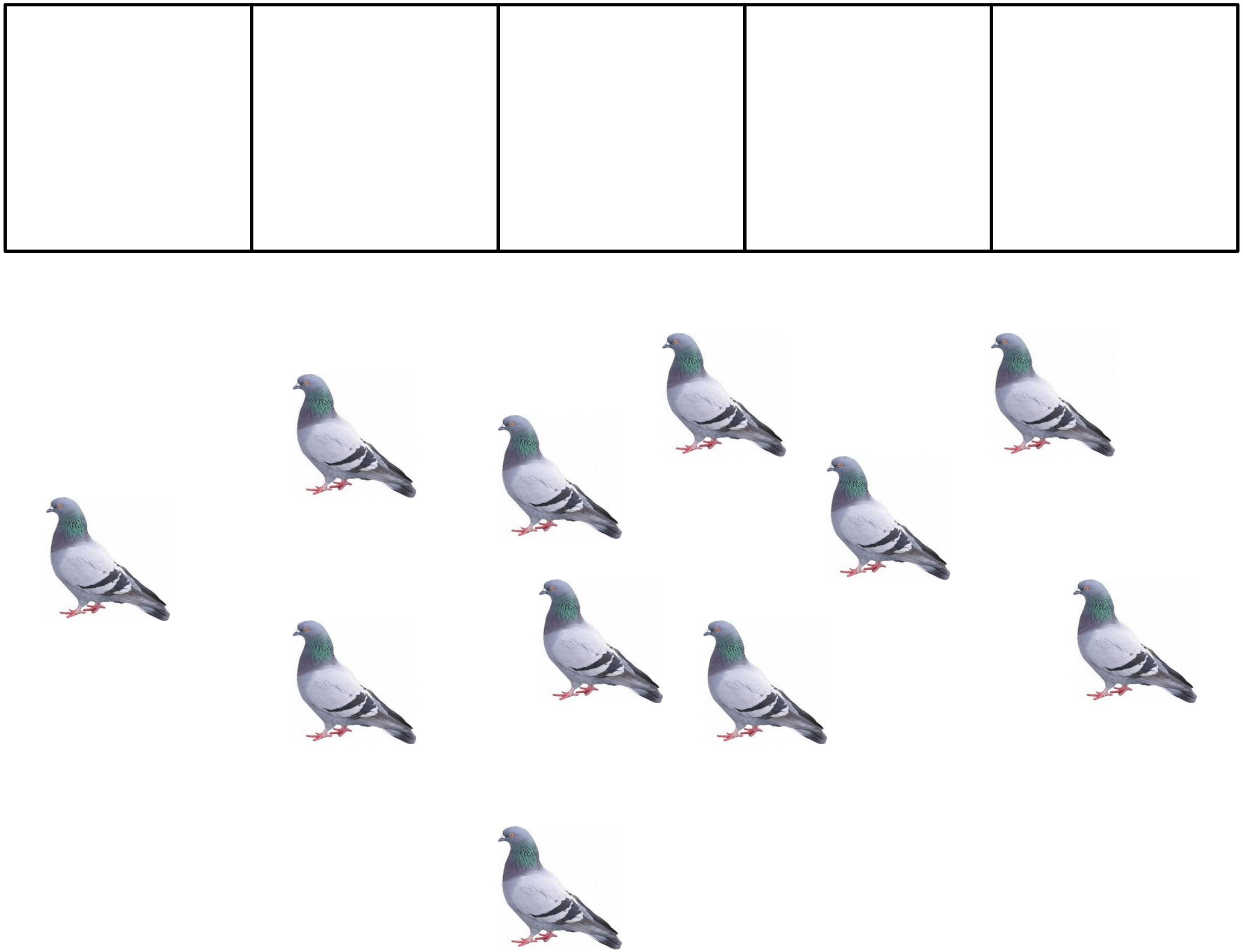
Theorem: In any graph with at least two nodes, there are at least two nodes of the same degree.

Proof 2: Assume for the sake of contradiction that there is a graph G with $n \geq 2$ nodes where no two nodes have the same degree. There are n possible choices for the degrees of nodes in G, namely $0, 1, 2, \ldots, n - 1$, so this means that G must have exactly one node of each degree. However, this means that G has a node of degree 0 and a node of degree $n - 1$. (These can't be the same node, since $n \geq 2$.) This first node is adjacent to no other nodes, but this second node is adjacent to every other node, which is impossible.

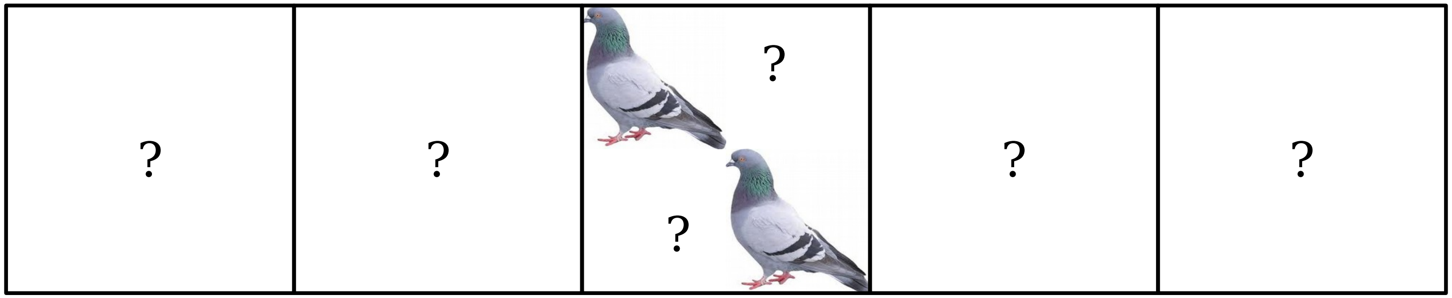
We have reached a contradiction, so our assumption must have been wrong. Thus if G is a graph with at least two nodes, G must have at least two nodes of the same degree. ■

The Generalized Pigeonhole Principle

The Pigeonhole Principle



The Pigeonhole Principle



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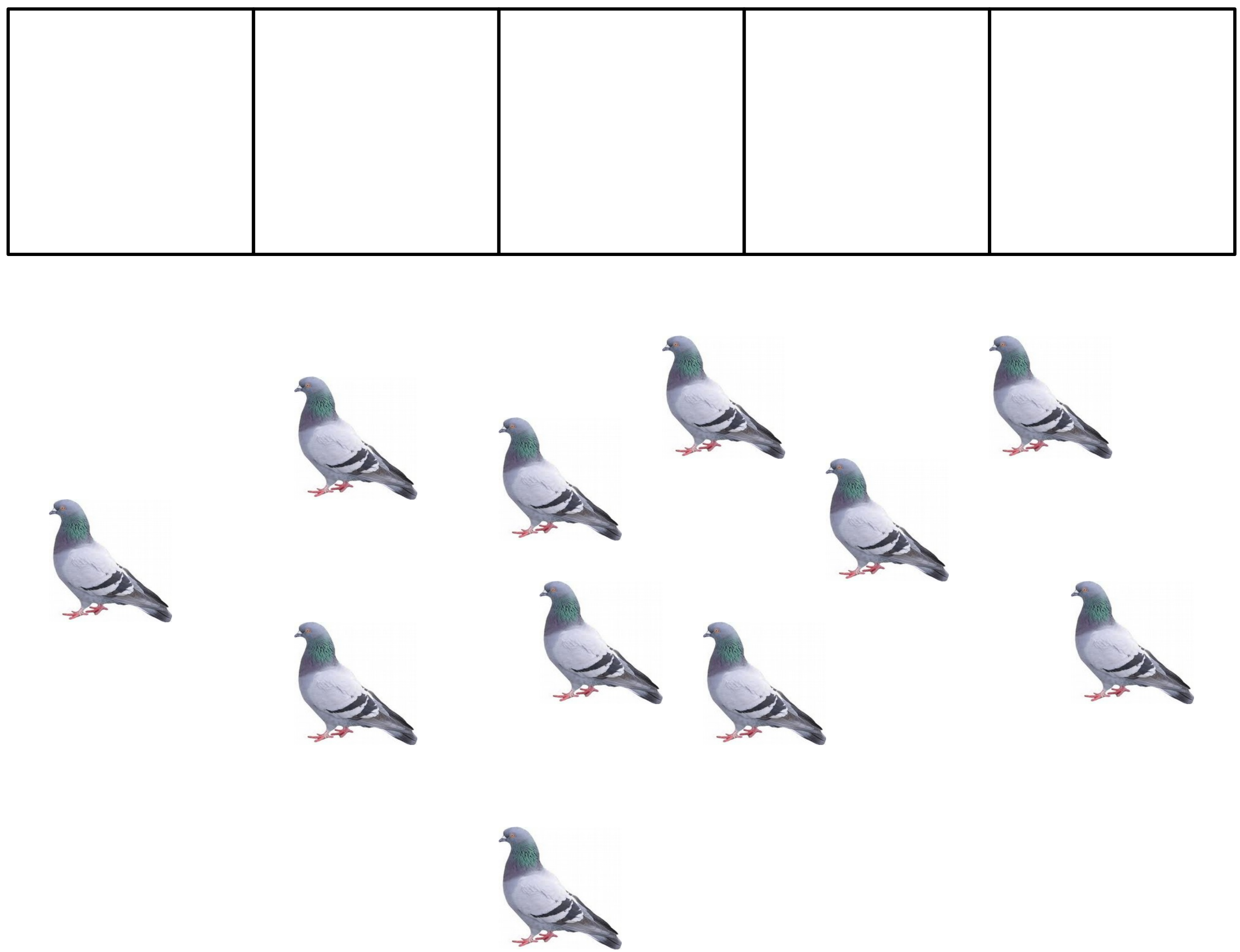

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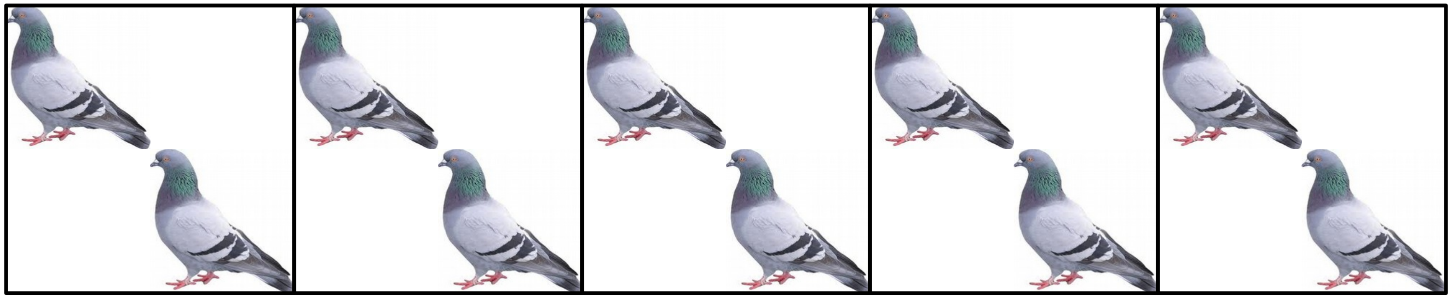

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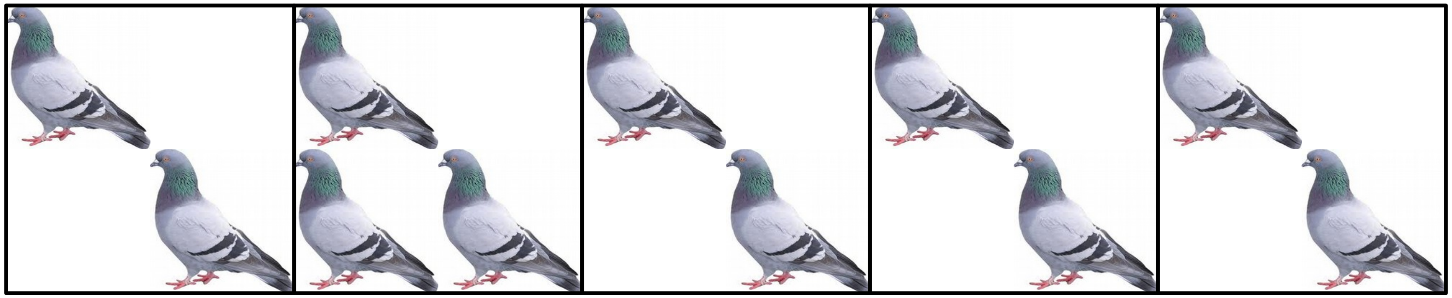

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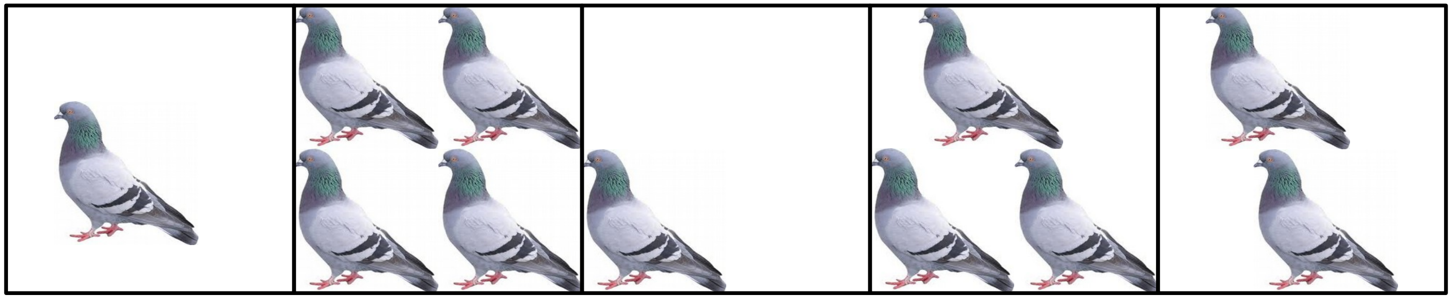
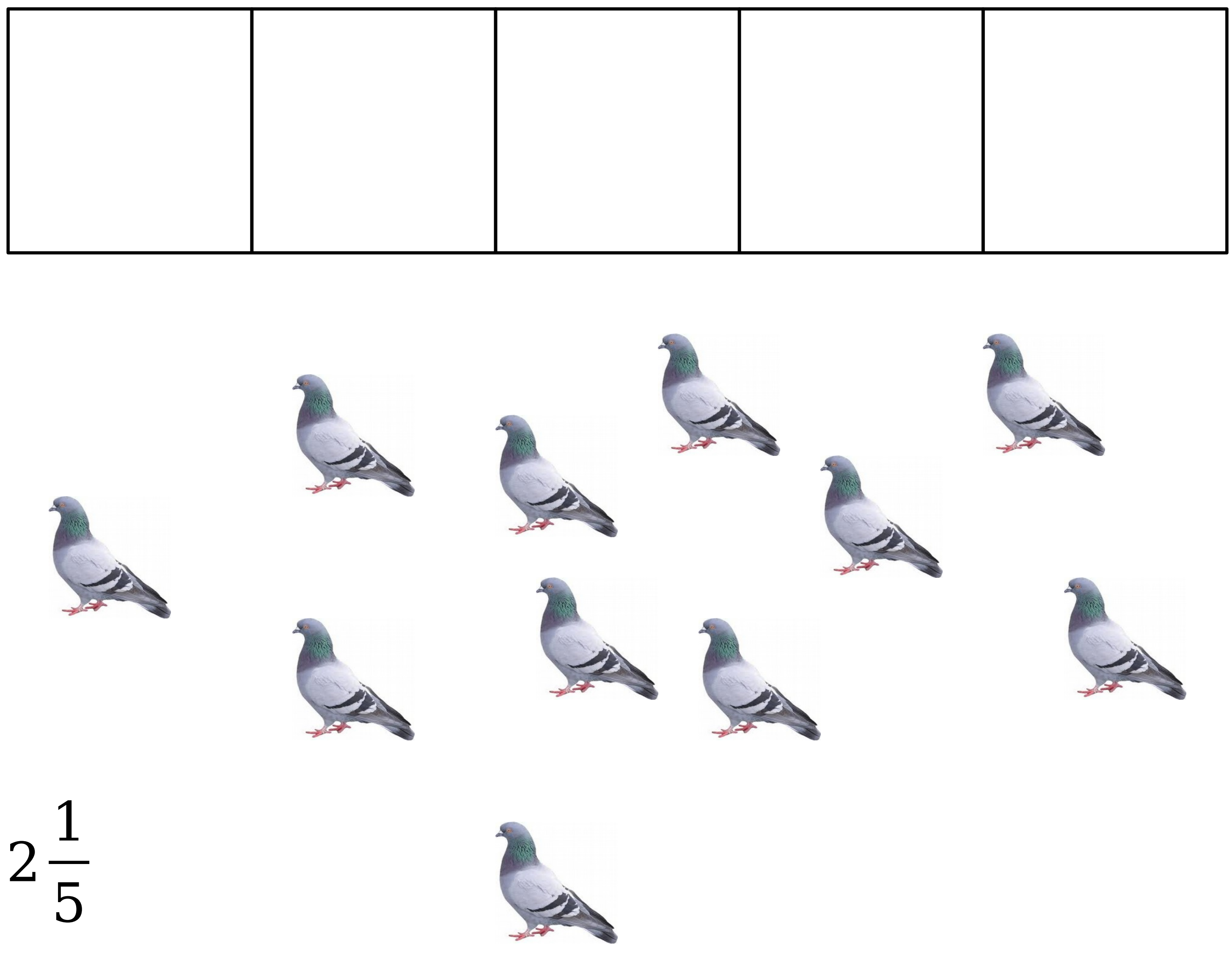


Imagine you trying to put 11 objects into 5 bins. How many of the following statements are true?

- The bin with the most objects must contain at least 2 objects.
- The bin with the most objects must contain at least 3 objects.
- The bin with the most objects must contain at least 4 objects.
- The bin with the fewest objects must contain at most 1 object.
- The bin with the fewest objects must contain at most 2 objects.
- The bin with the fewest objects must contain at most 3 objects.

Respond at pollev.com/zhenglian740

The Pigeonhole Principle

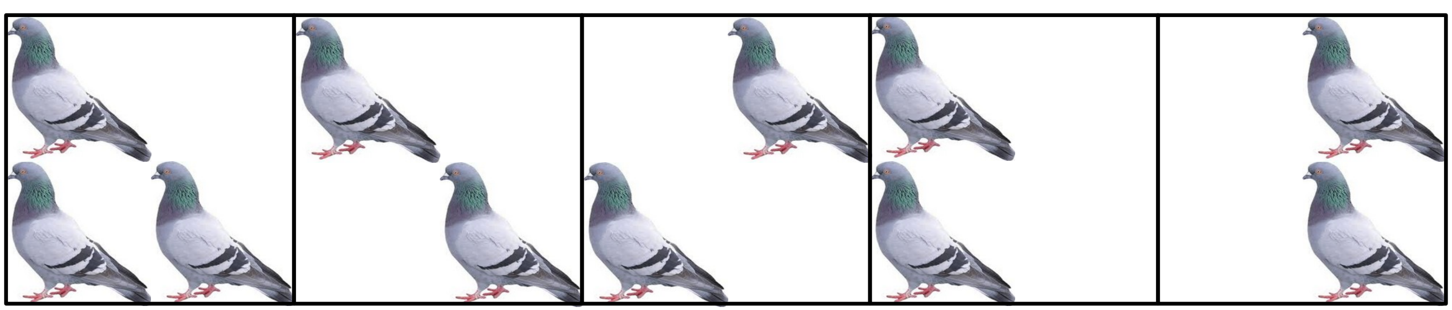


$$\frac{11}{5} = 2\frac{1}{5}$$

A More General Version

- The ***generalized pigeonhole principle*** says that if you distribute *m* objects into *n* bins, then
  - some bin will have at least $\lceil m/n \rceil$ objects in it, and
  - some bin will have at most $\lfloor m/n \rfloor$ objects in it.

$\lceil m/n \rceil$ means "m/n, rounded up."
$\lfloor m/n \rfloor$ means "m/n, rounded down."



$m = 11$
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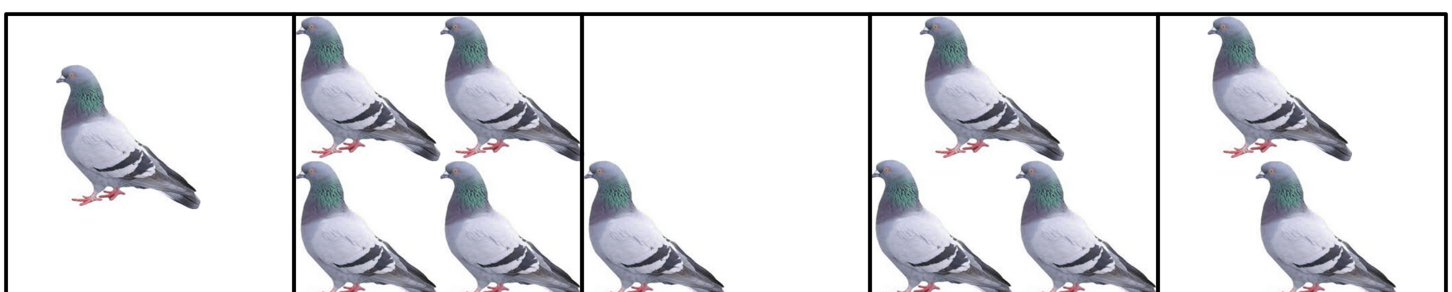

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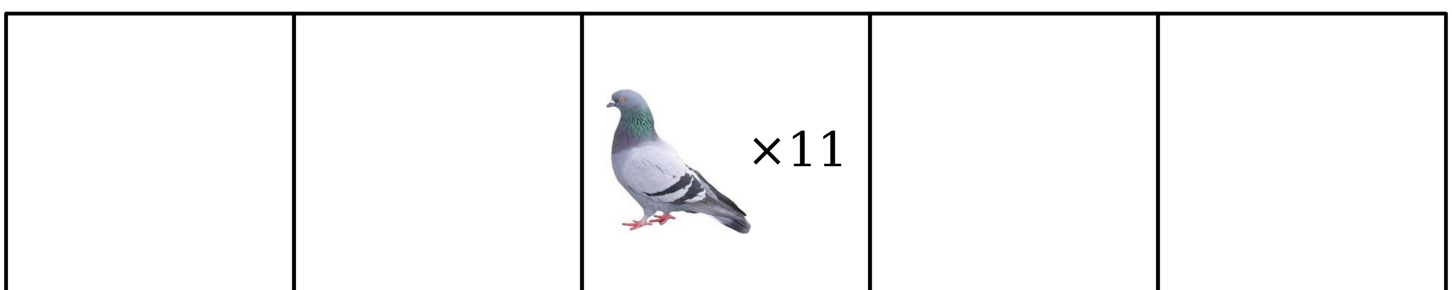

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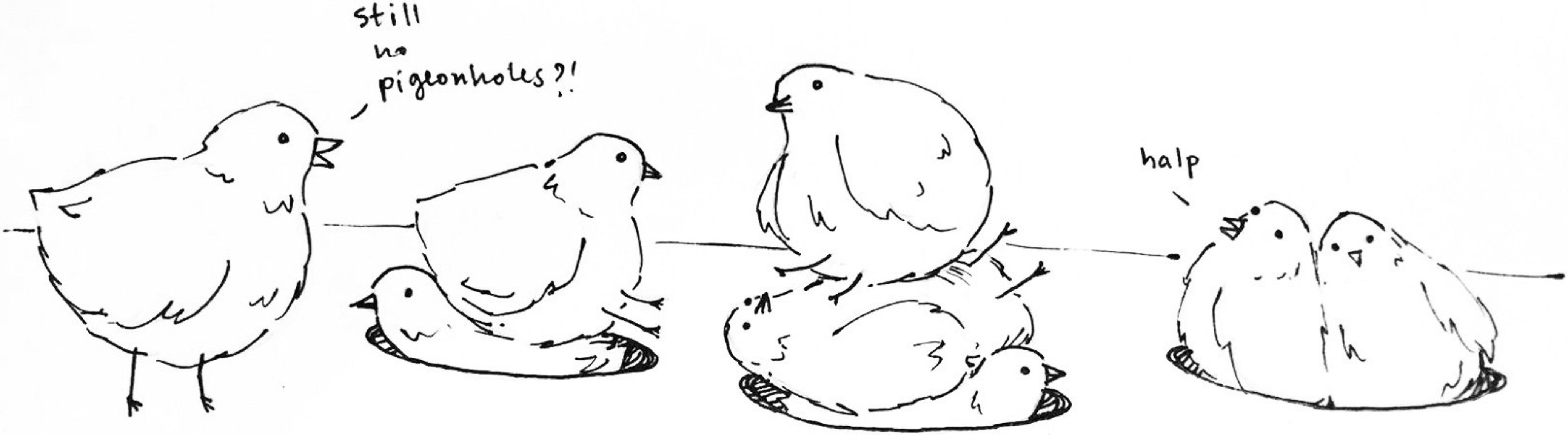


$m = 8, n = 3$

Theorem: If m objects are distributed into $n > 0$ bins, then some bin will contain at least $\lceil m/n \rceil$ objects.

Proof: We will prove that if m objects are distributed into n bins, then some bin contains at least m/n objects. Since the number of objects in each bin is an integer, this will prove that some bin must contain at least $\lceil m/n \rceil$ objects.

To do this, we proceed by contradiction. Suppose that, for some m and n, there is a way to distribute m objects into n bins such that each bin contains fewer than m/n objects.

Number the bins 1, 2, 3, …, n and let x_i denote the number of objects in bin i. Since there are m objects in total, we know that

$$m = x_1 + x_2 + \ldots + x_n.$$

Since each bin contains fewer than m/n objects, we see that $x_i < m/n$ for each i. Therefore, we have that

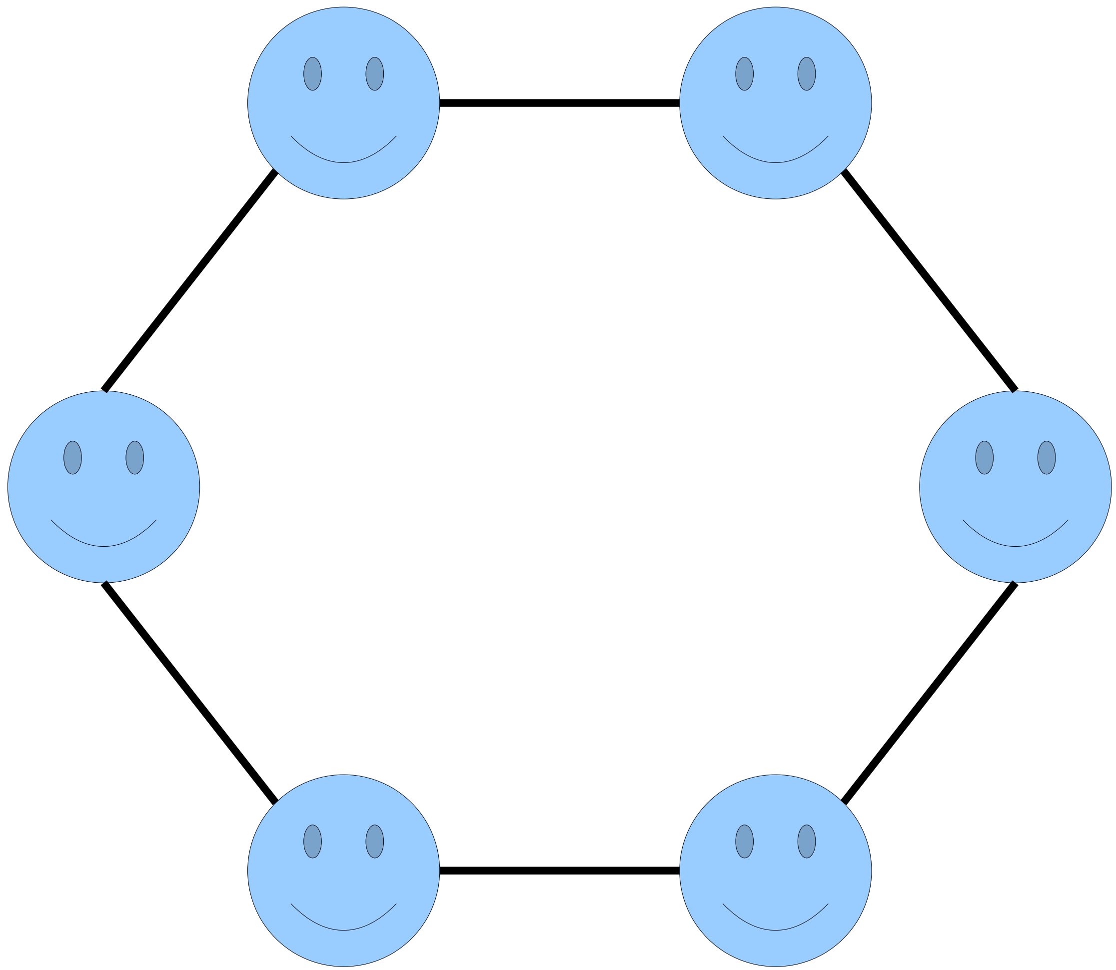
$$\begin{aligned} m &= x_1 + x_2 + \ldots + x_n \\ &< m/n + m/n + \ldots + m/n \quad (n \text{ times}) \\ &= m. \end{aligned}$$

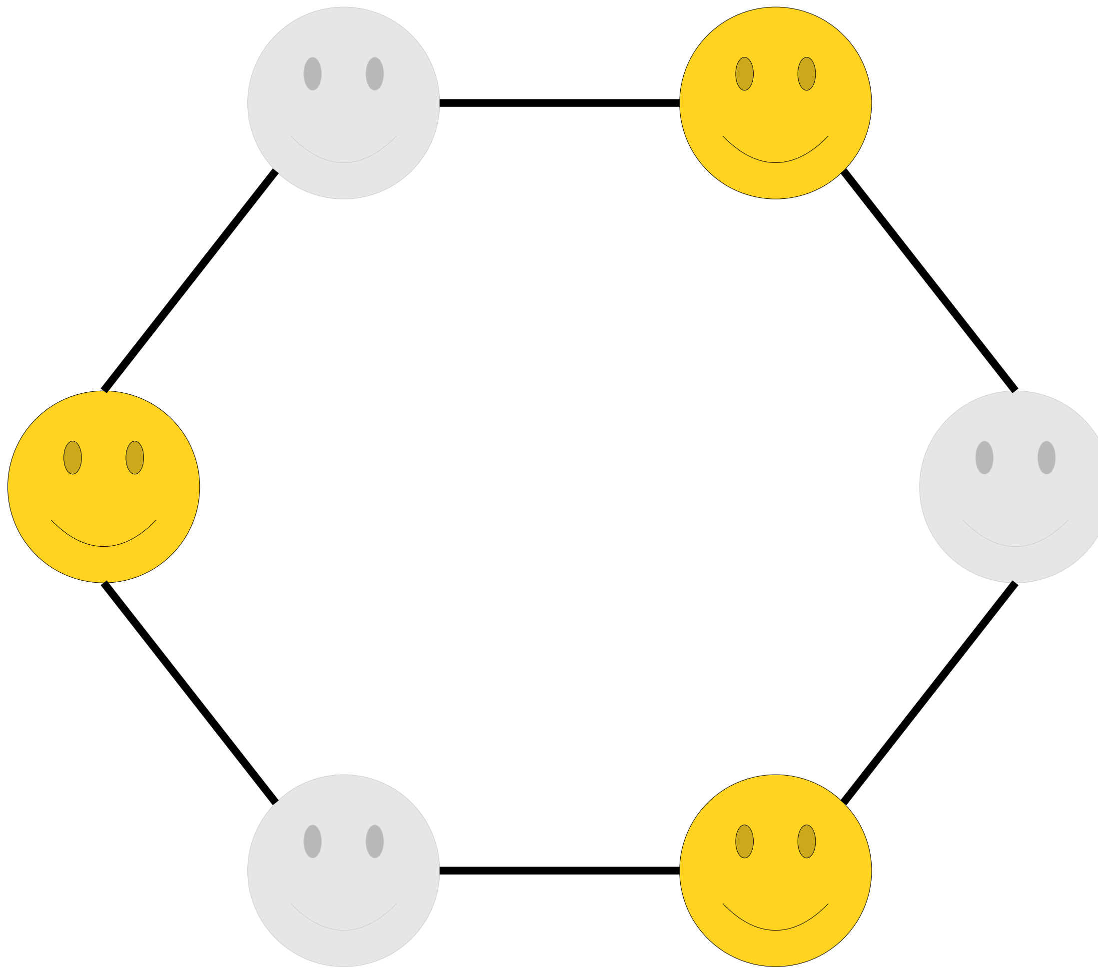
But this means that $m < m$, which is impossible. We have reached a contradiction, so our initial assumption must have been wrong. Therefore, if m objects are distributed into n bins, some bin must contain at least $\lceil m/n \rceil$ objects. ■

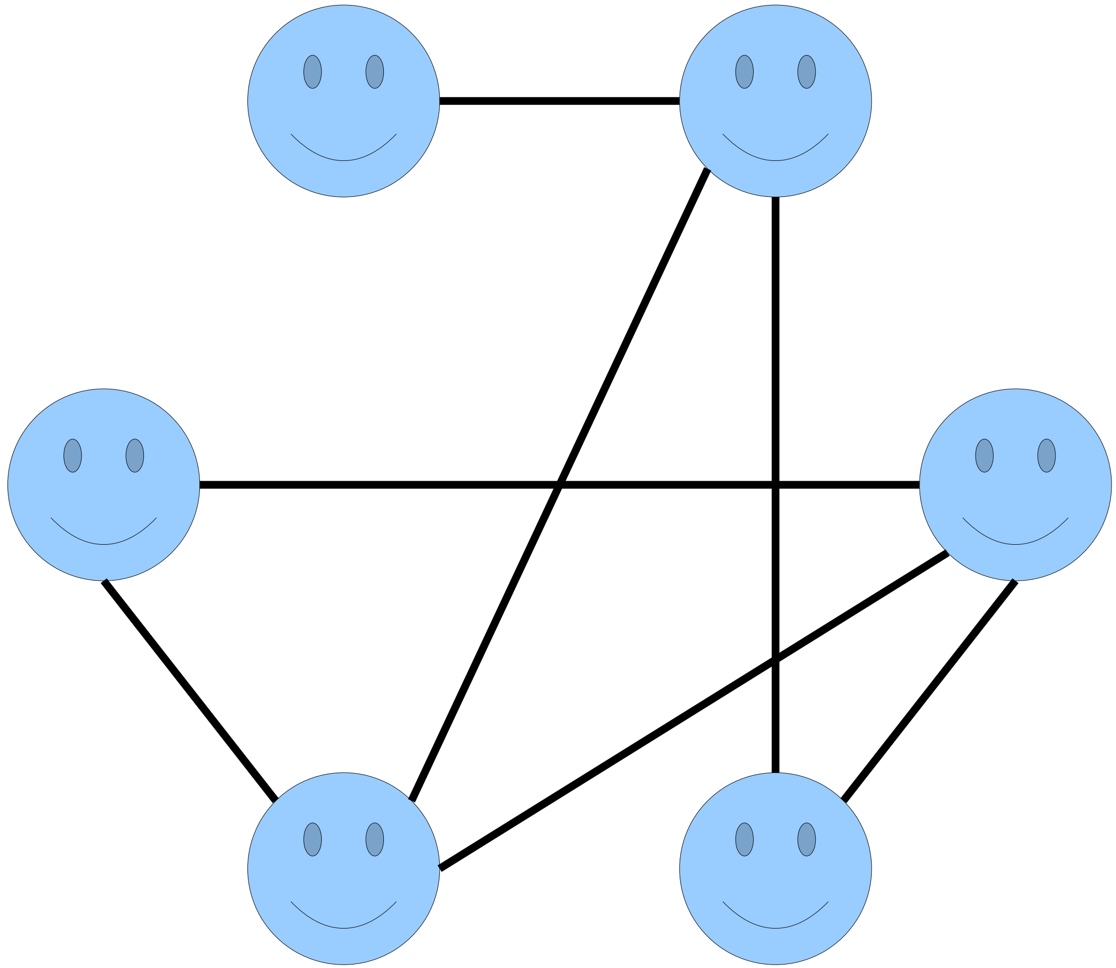
An Application: Friends and Strangers

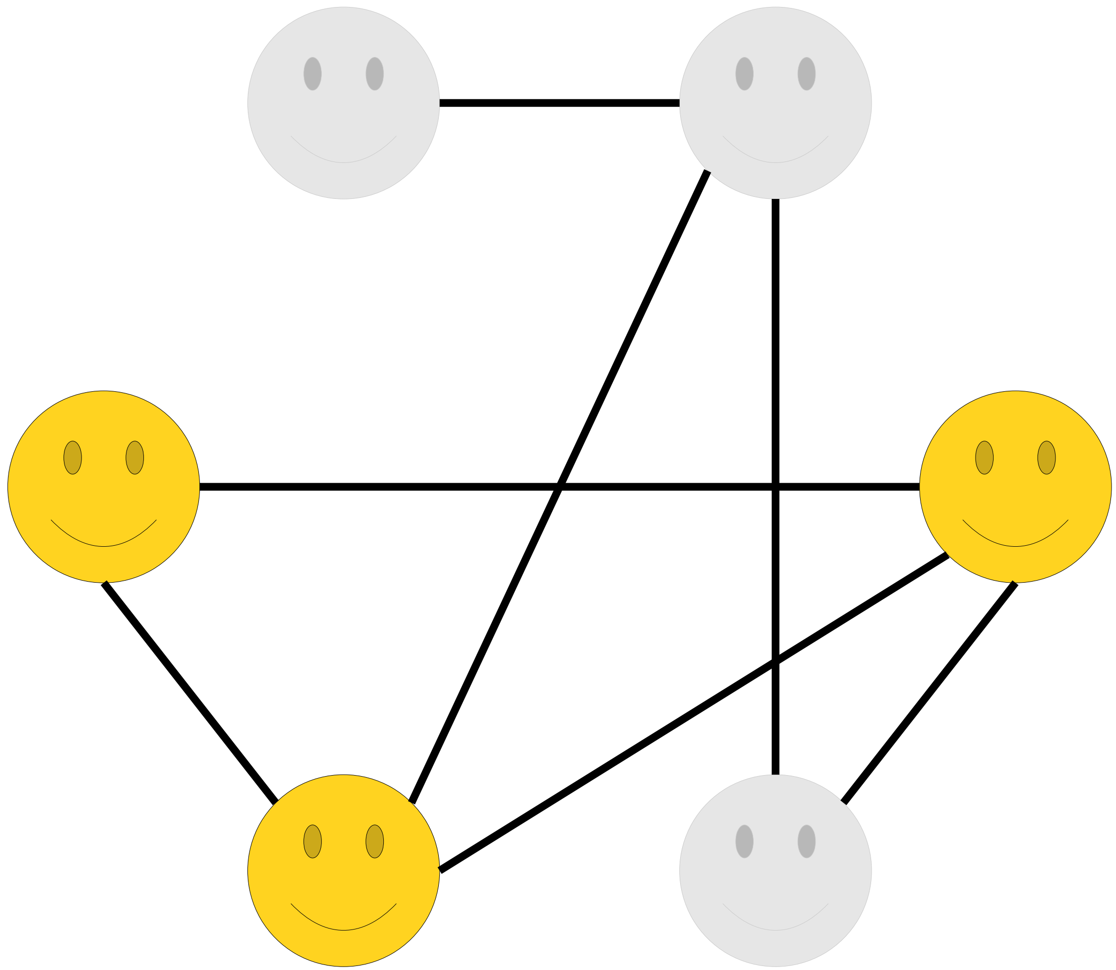
Friends and Strangers

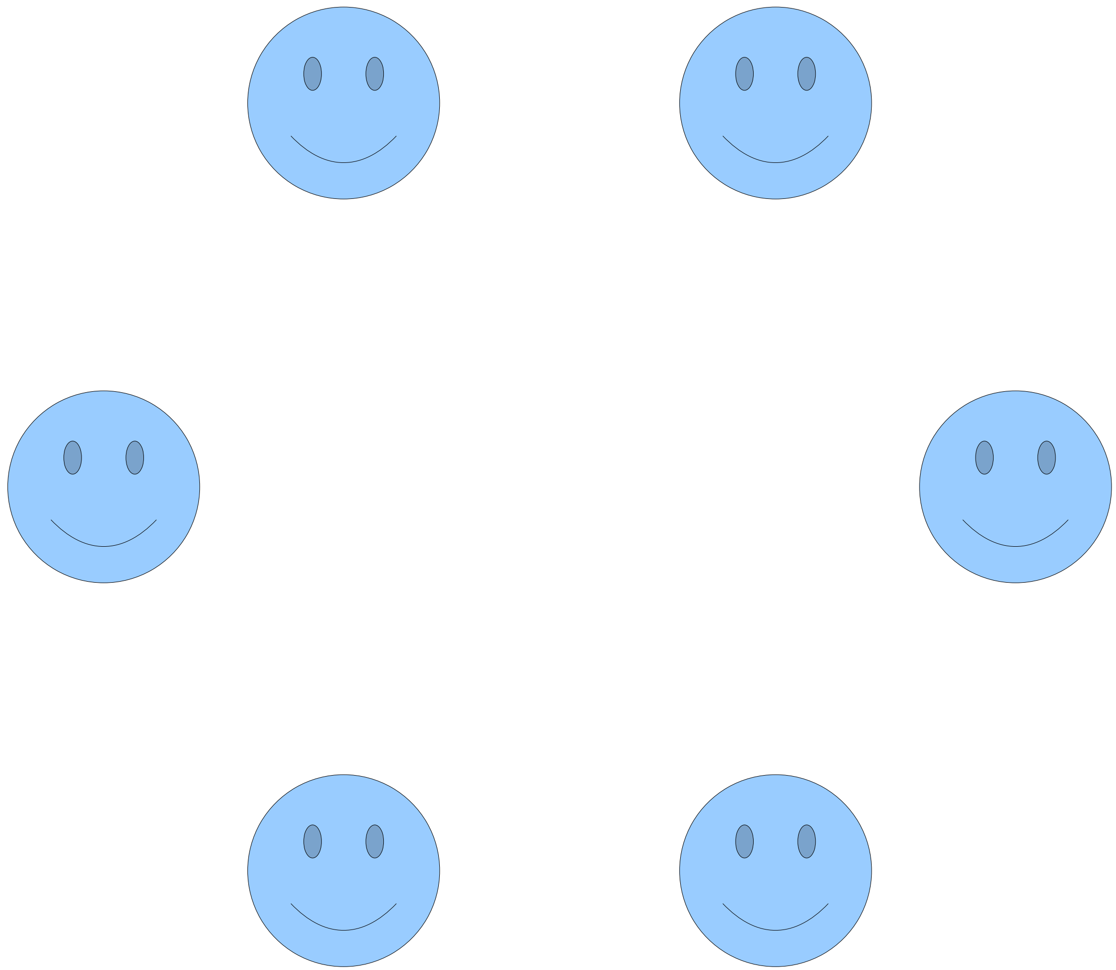
- Suppose you have a party of six people. Each pair of people are either friends (they know each other) or strangers (they do not).
- ***Theorem:*** Any such party must have a group of three mutual friends (three people who all know one another) or three mutual strangers (three people, none of whom know any of the others).

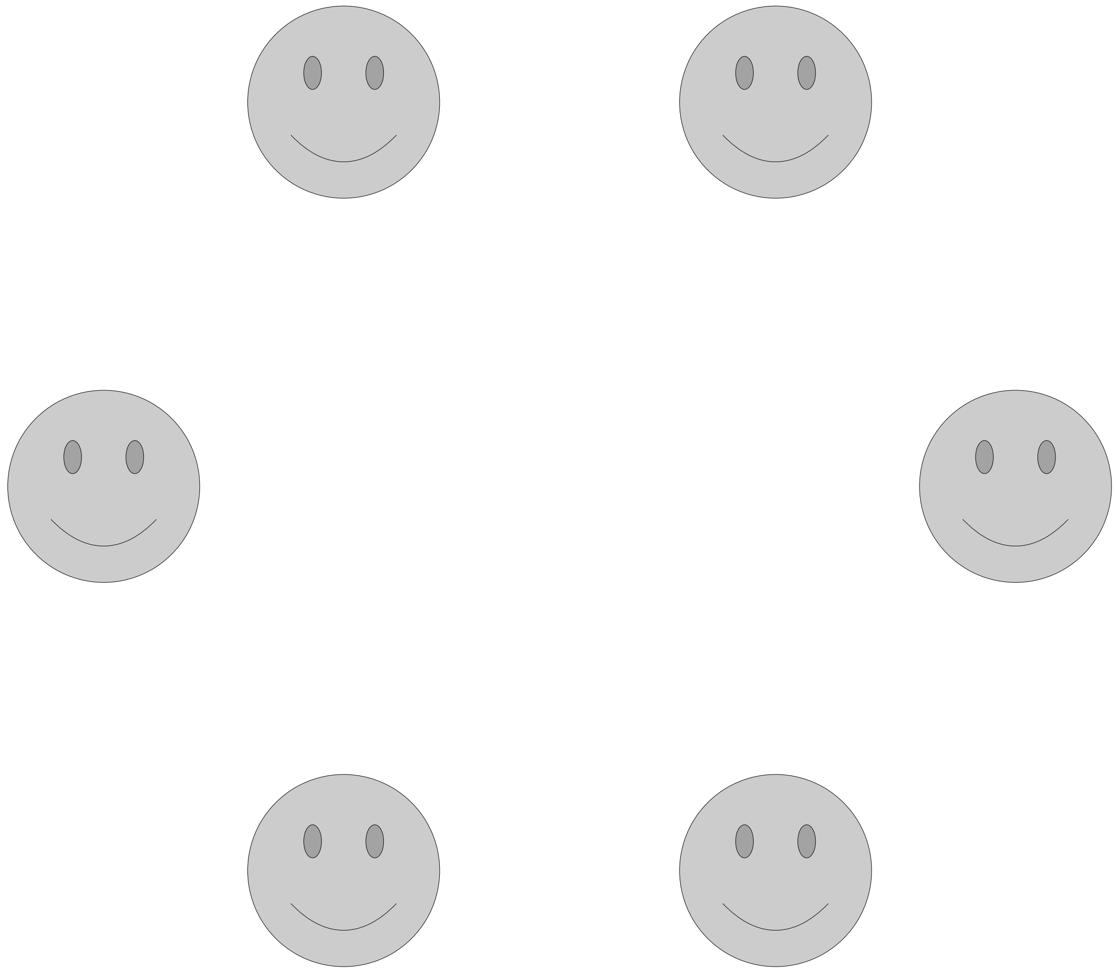


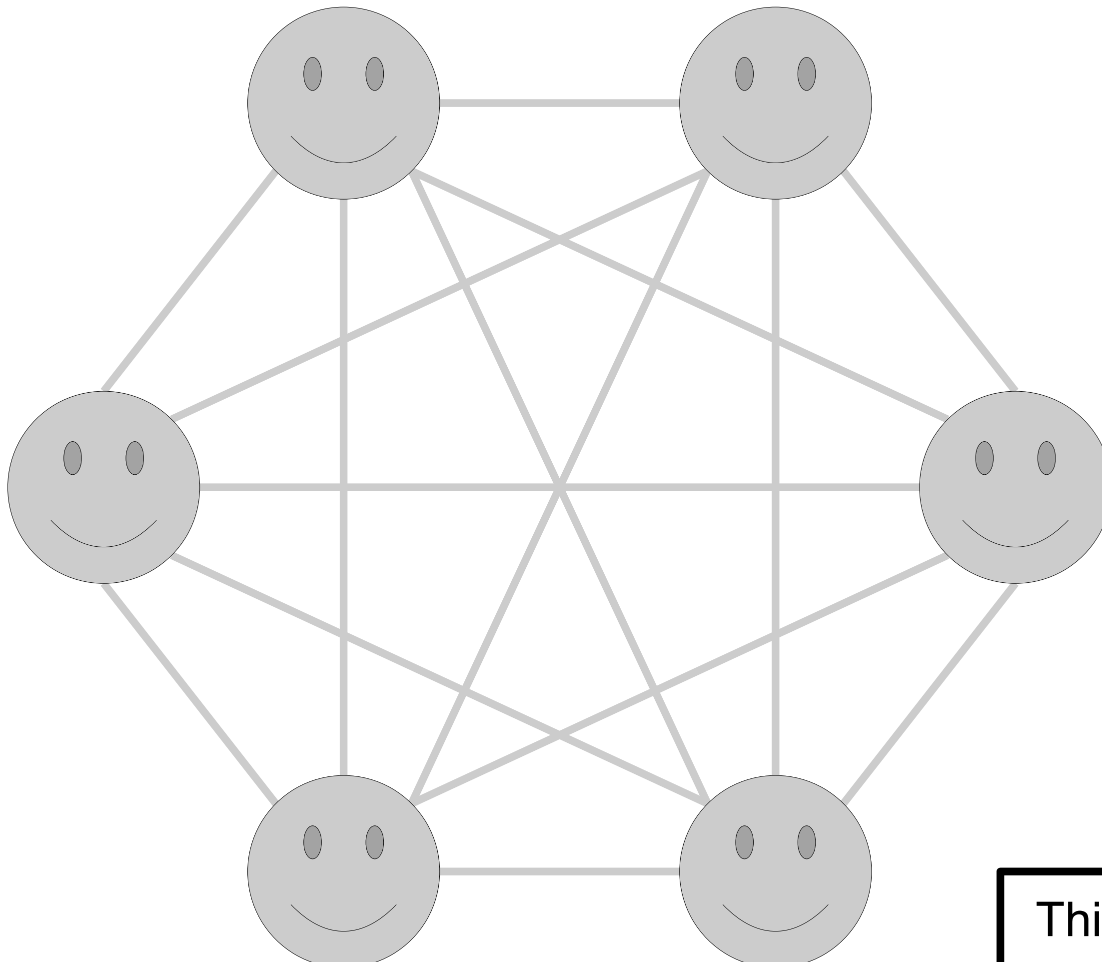




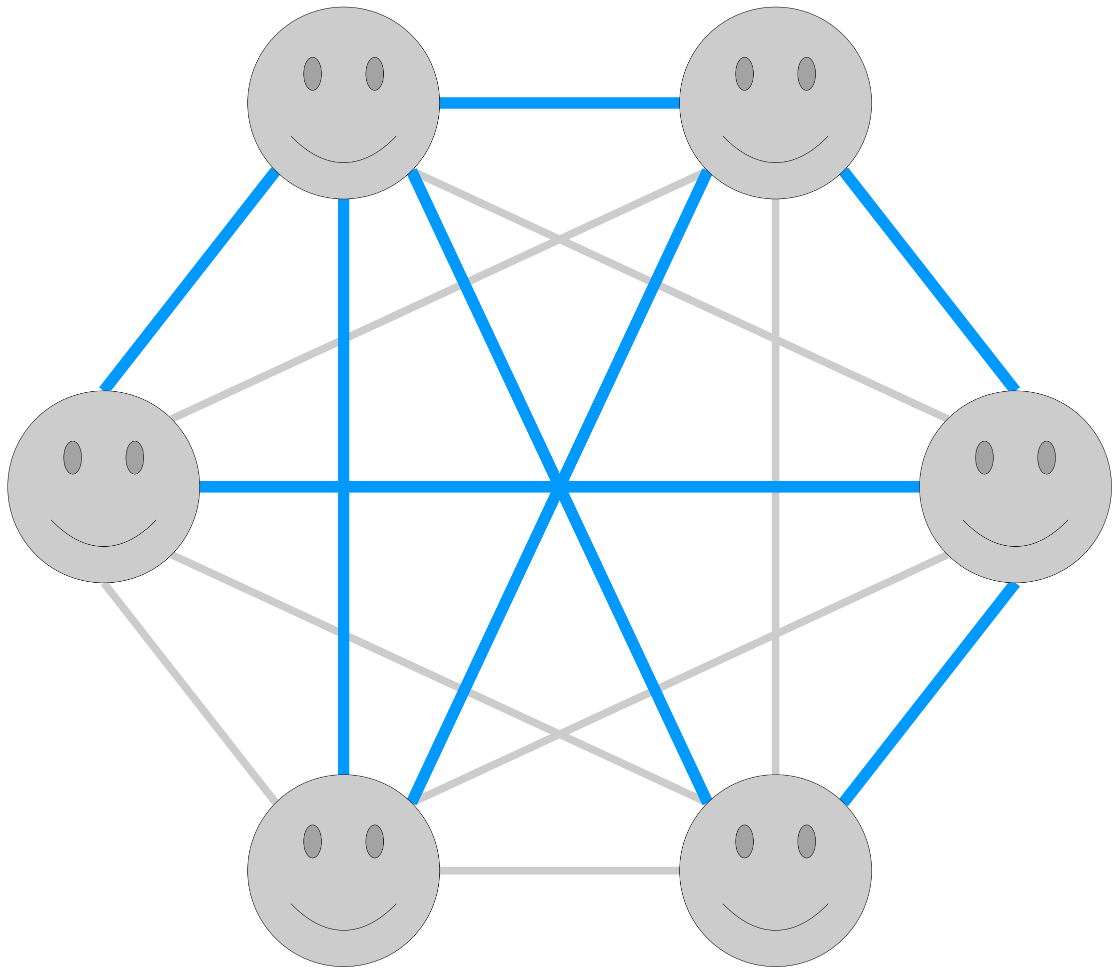


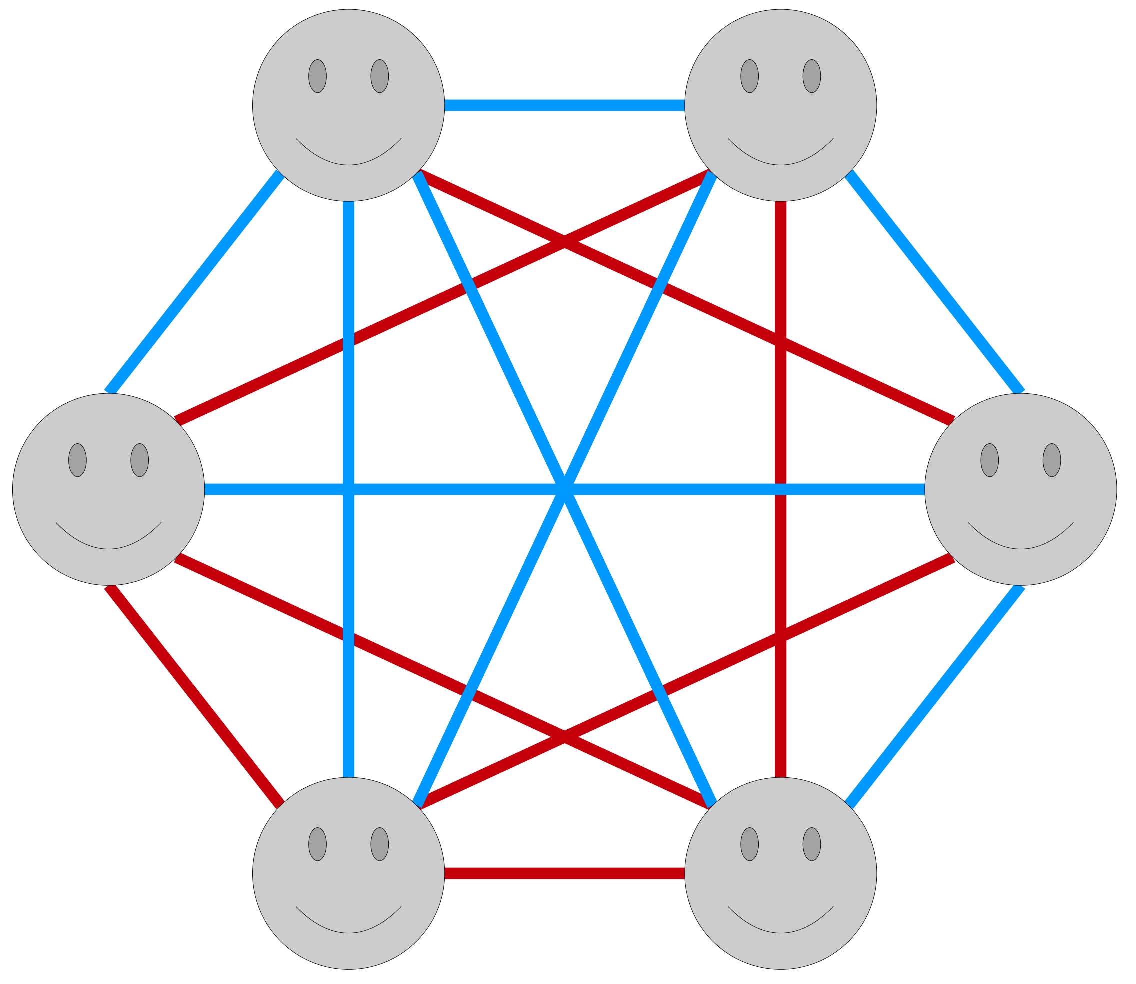


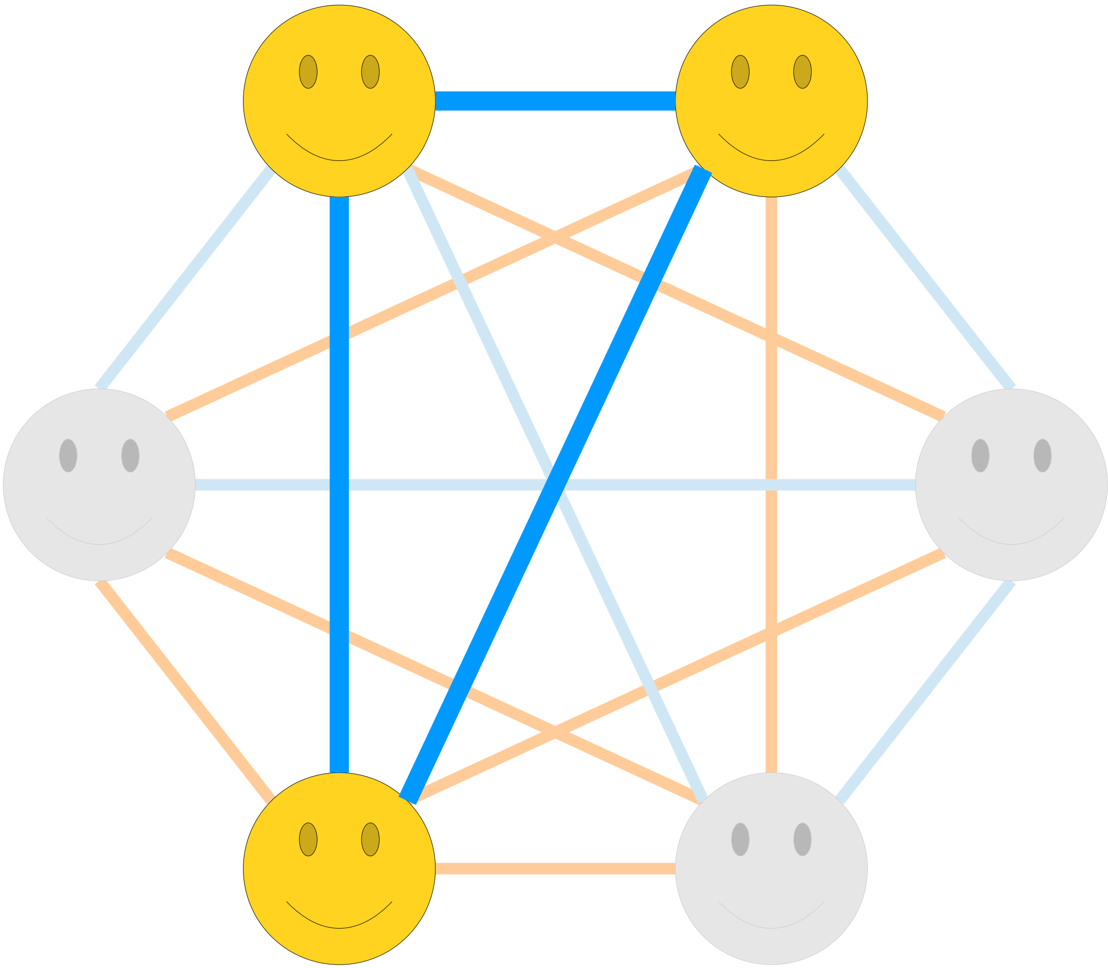


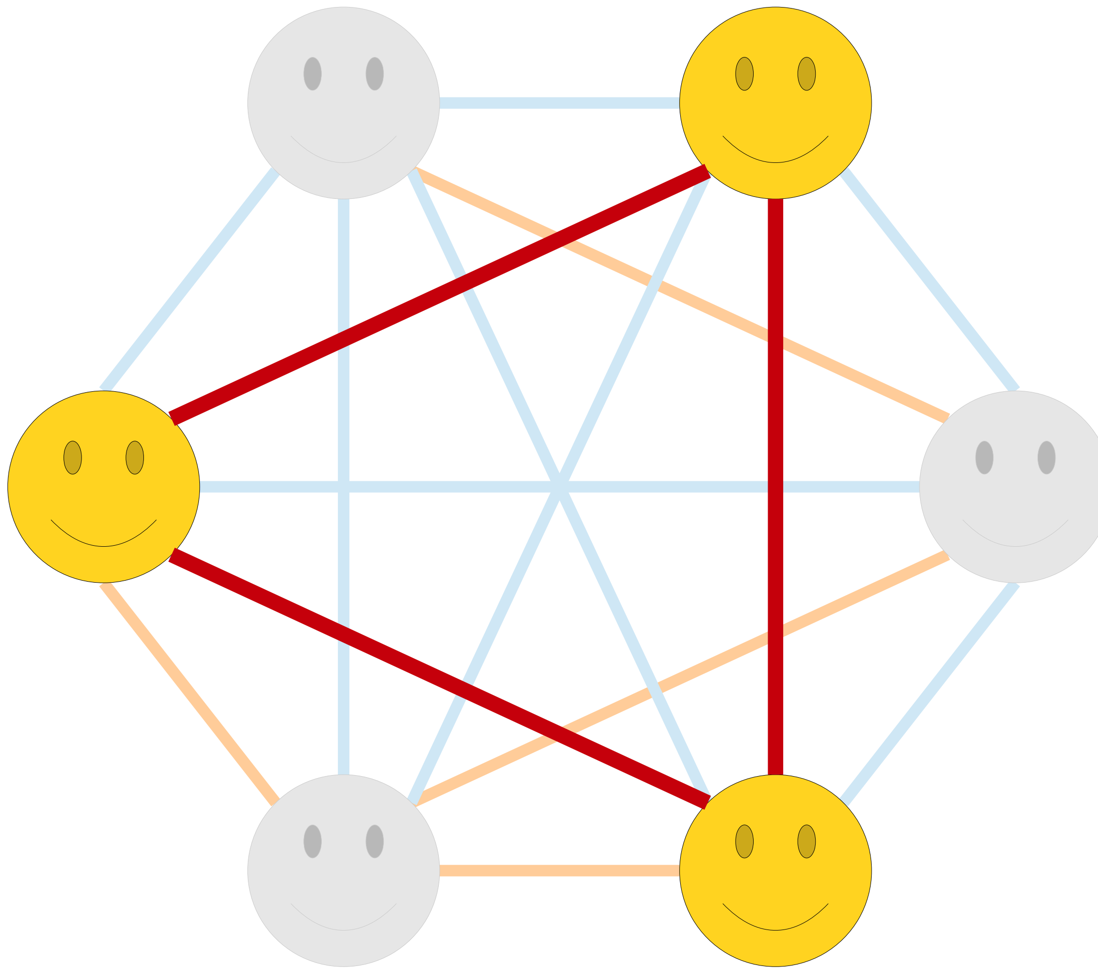


This graph is called a ***6-clique***, by the way.







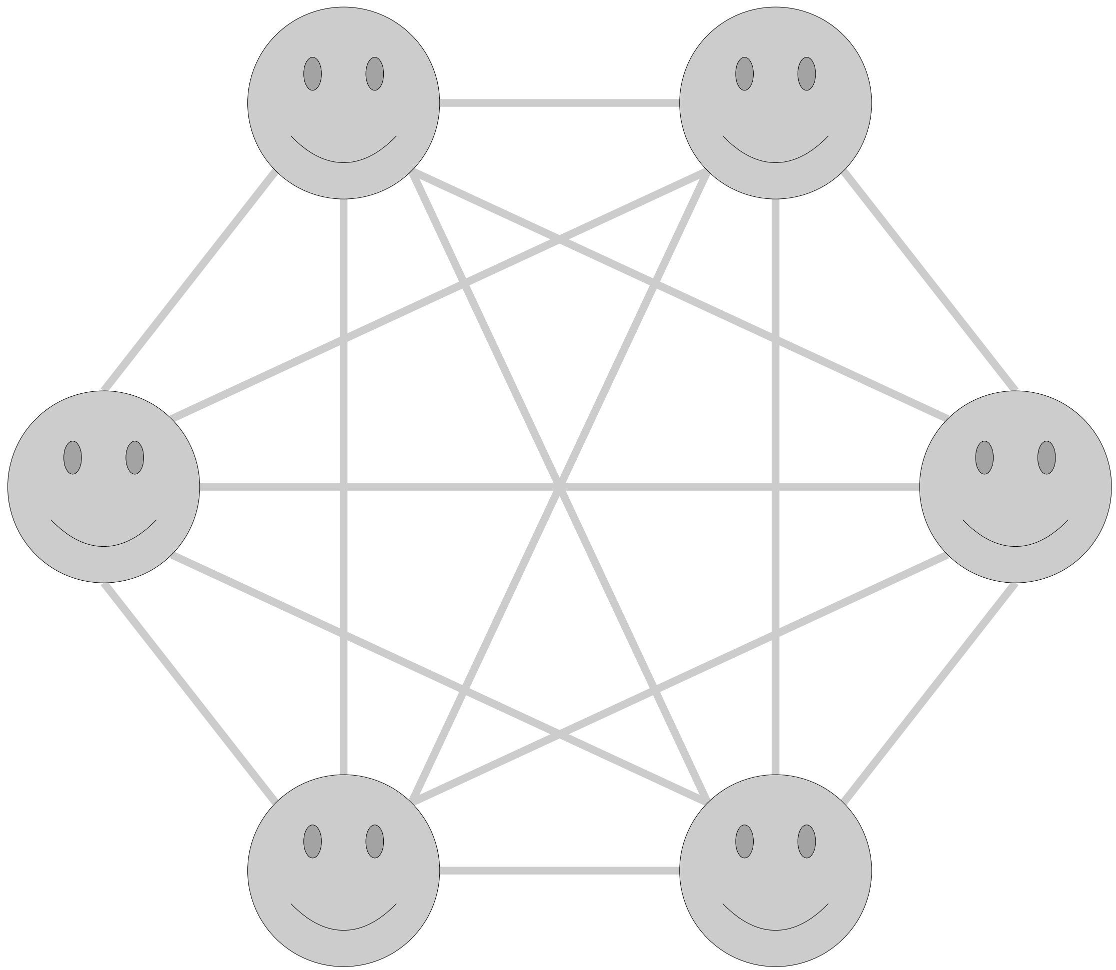


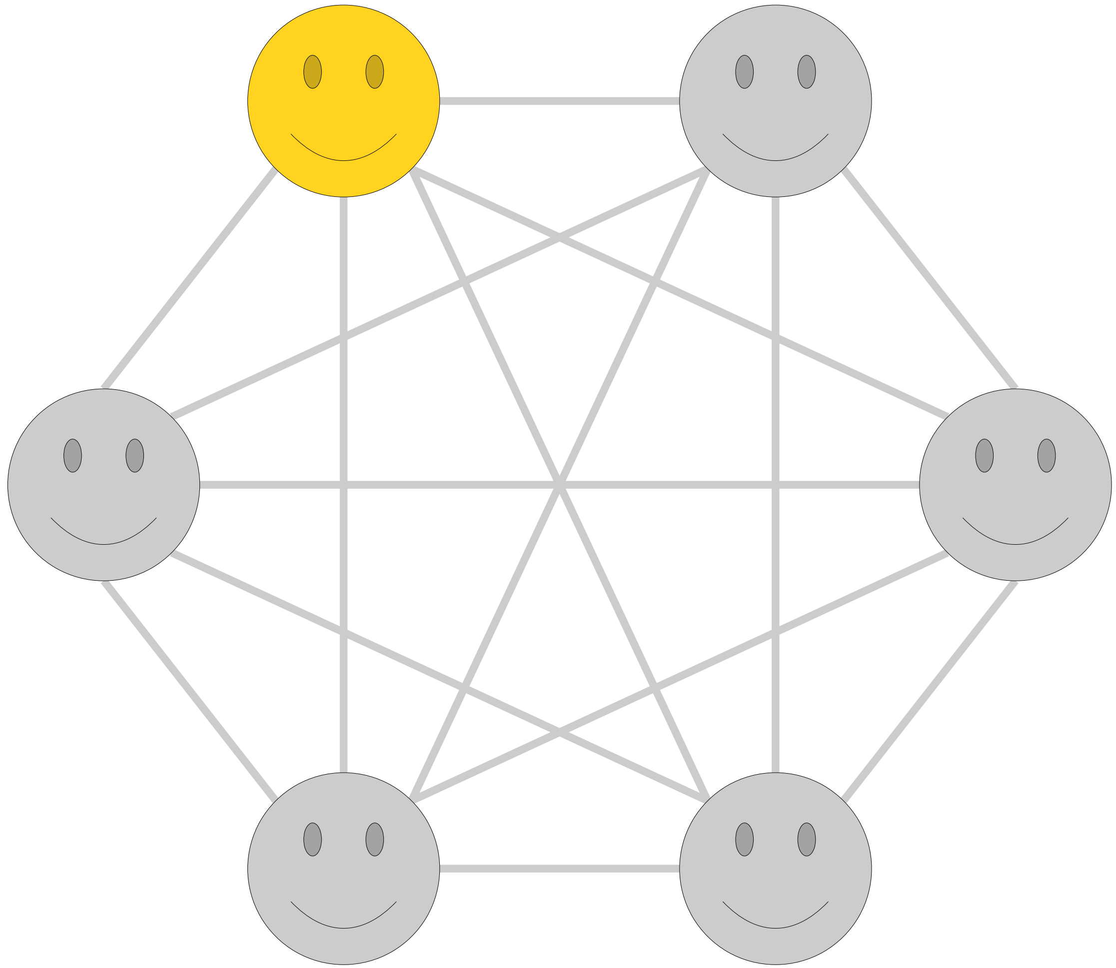
Friends and Strangers Restated

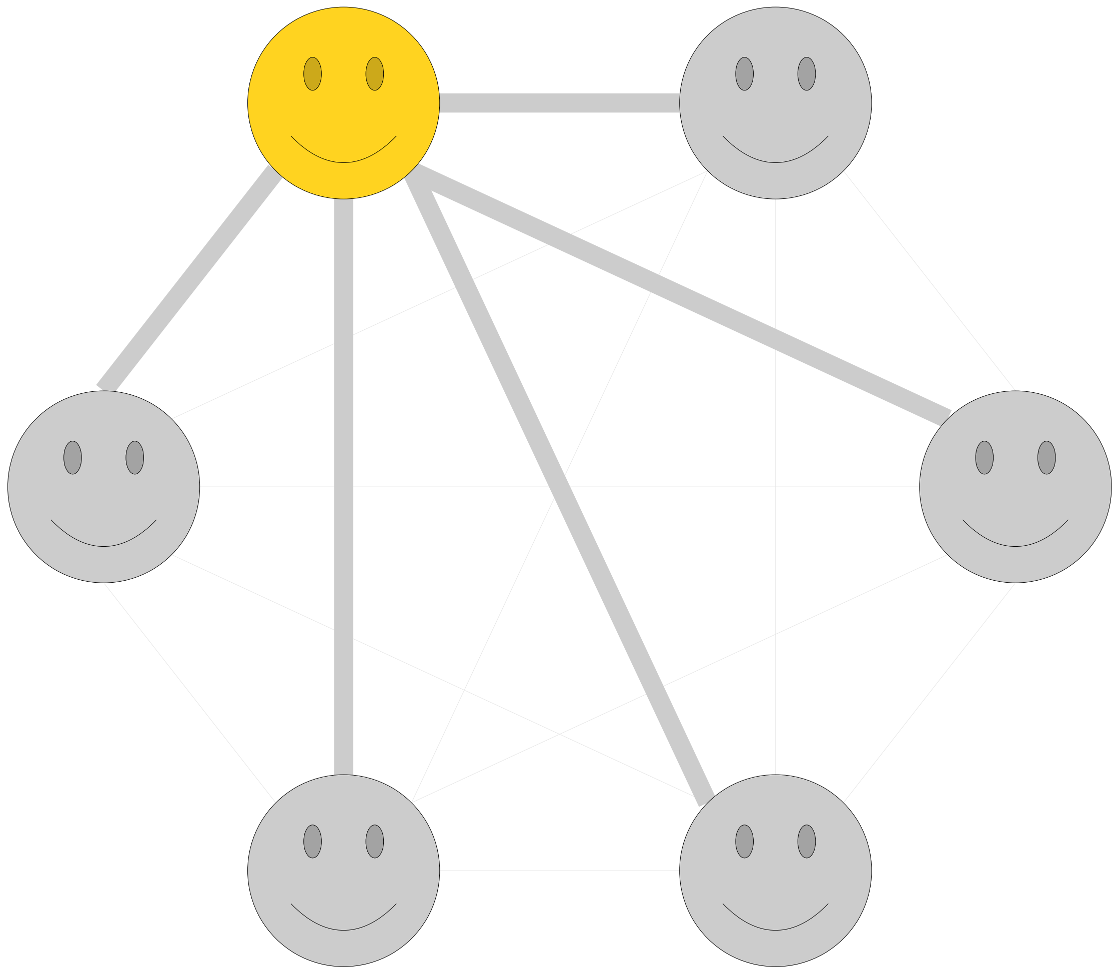
- From a graph-theoretic perspective, the Theorem on Friends and Strangers can be restated as follows:

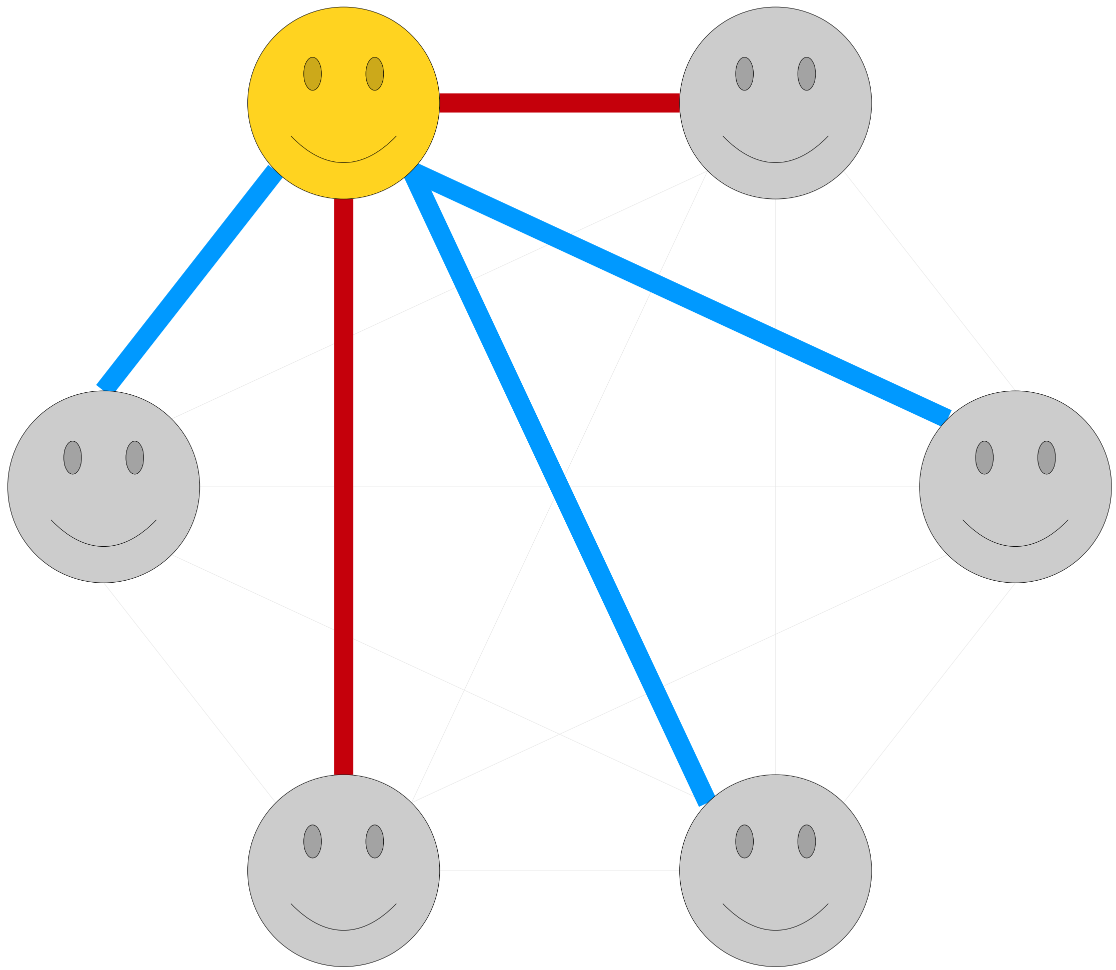
  Theorem: Any 6-clique whose edges are colored red and blue contains a red triangle or a blue triangle (or both).

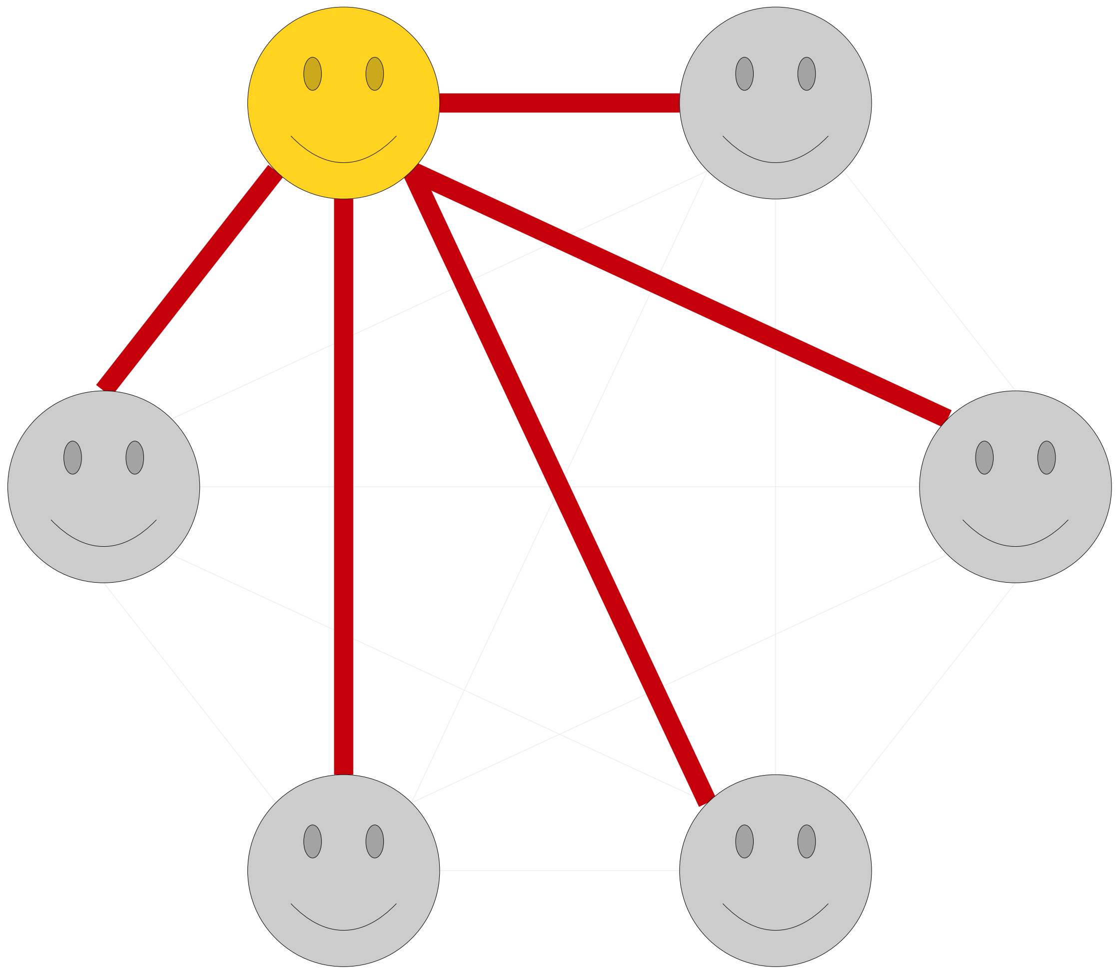
- How can we prove this?

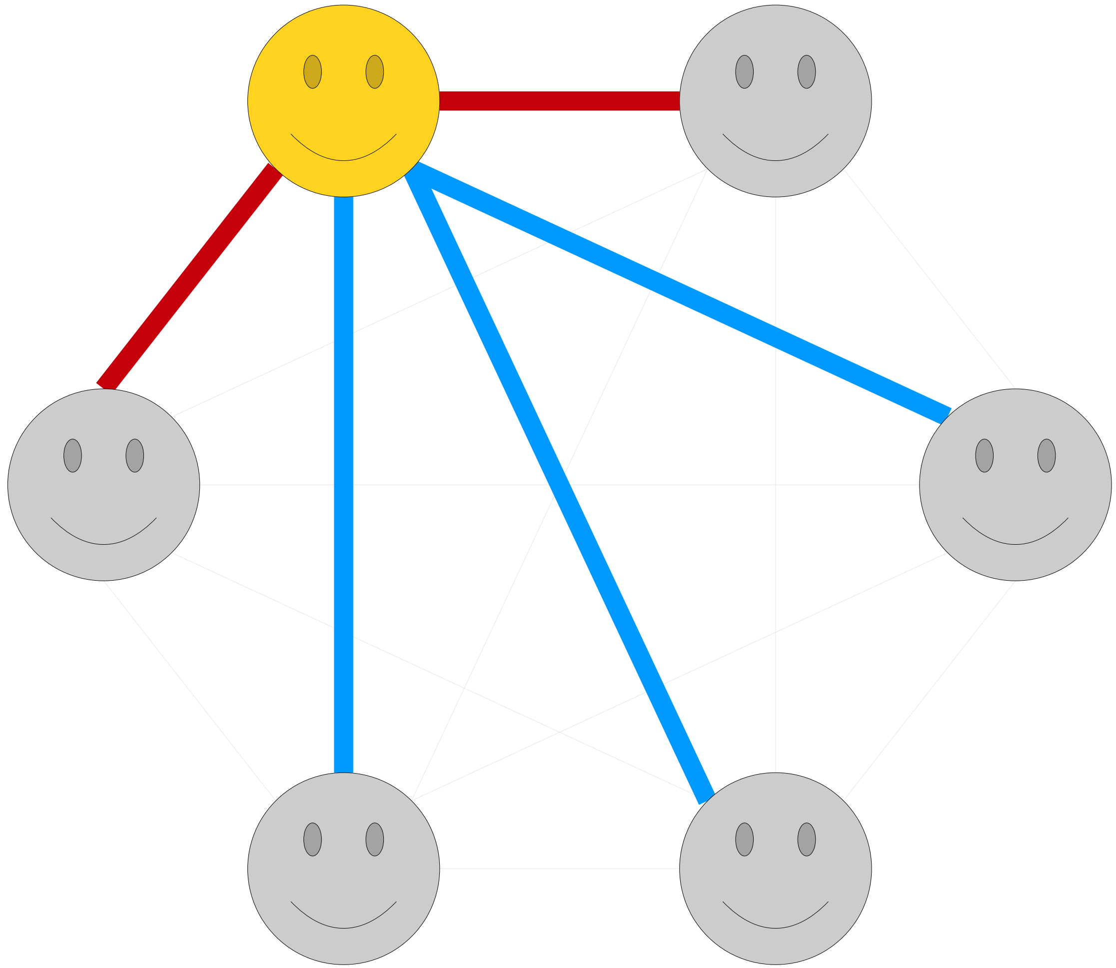


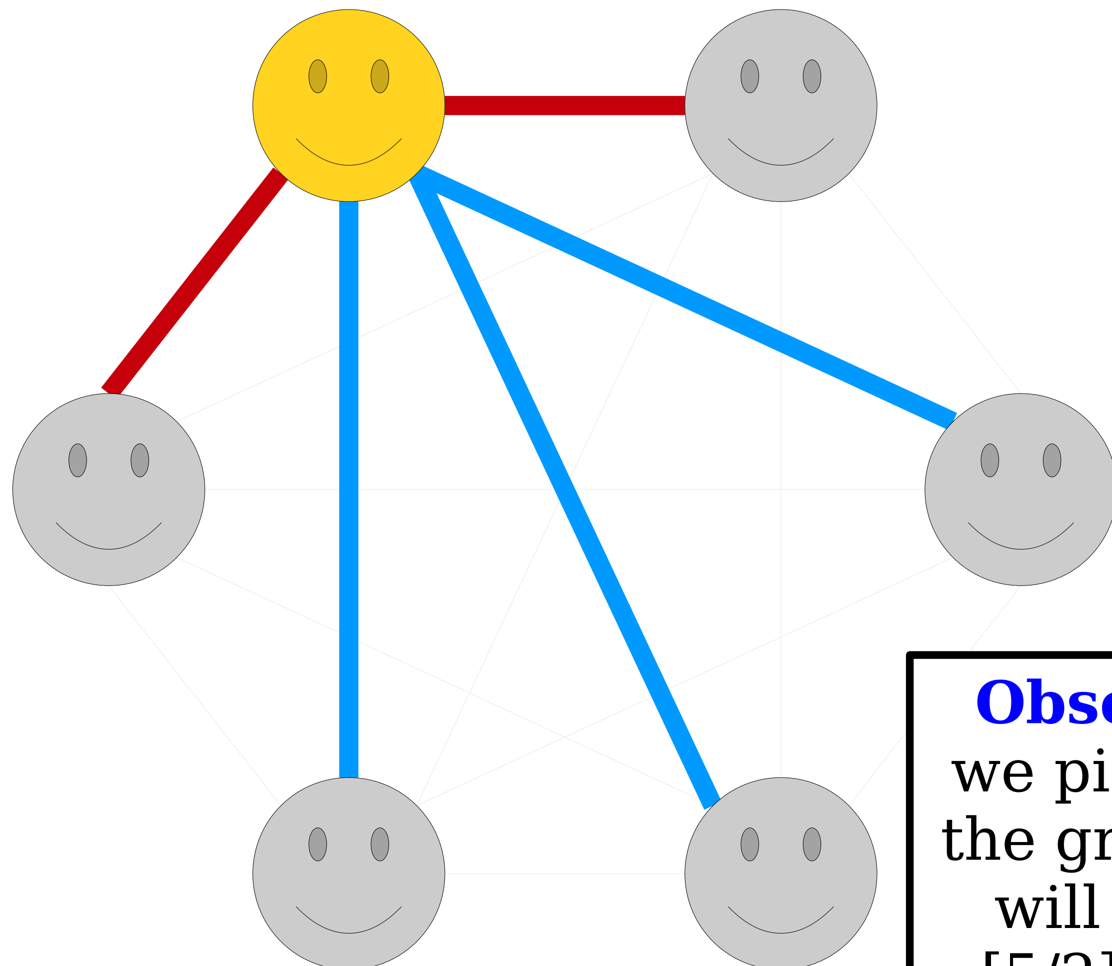




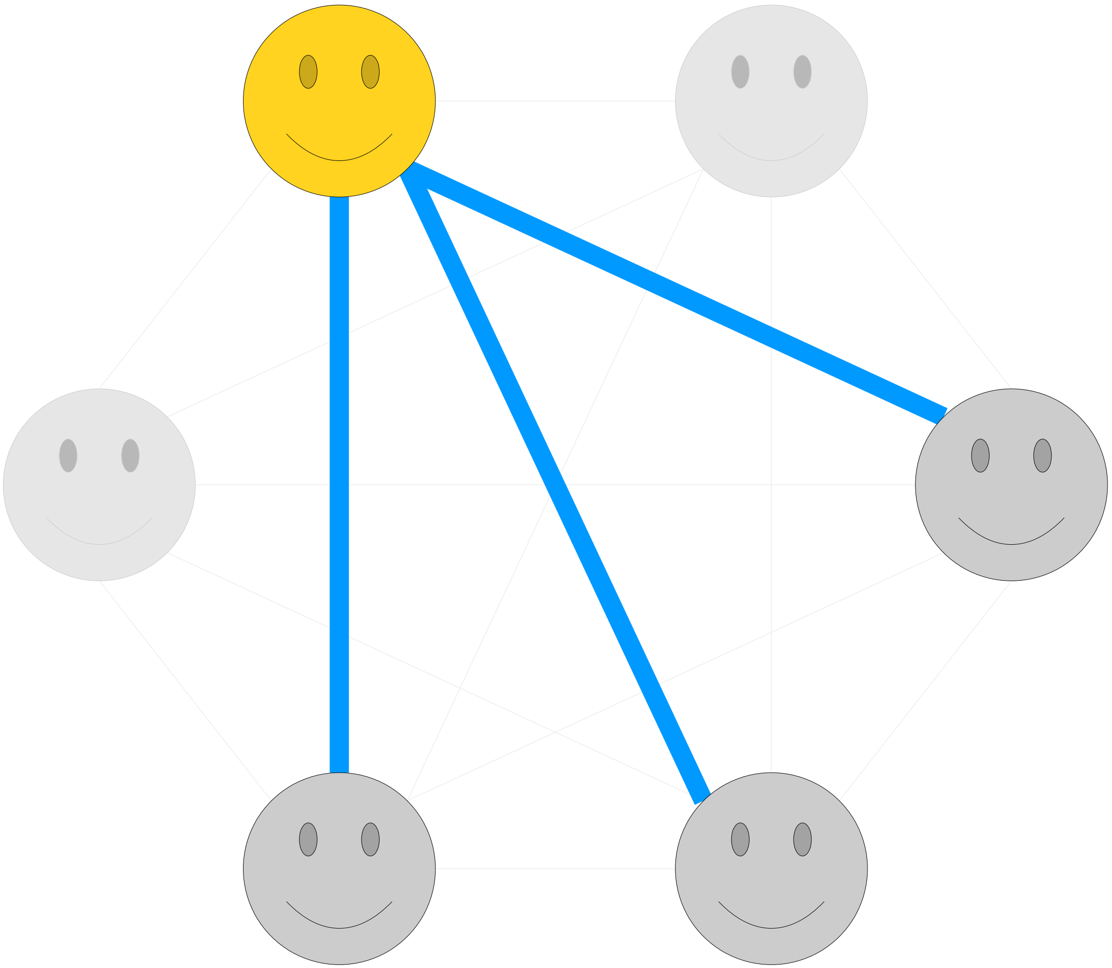


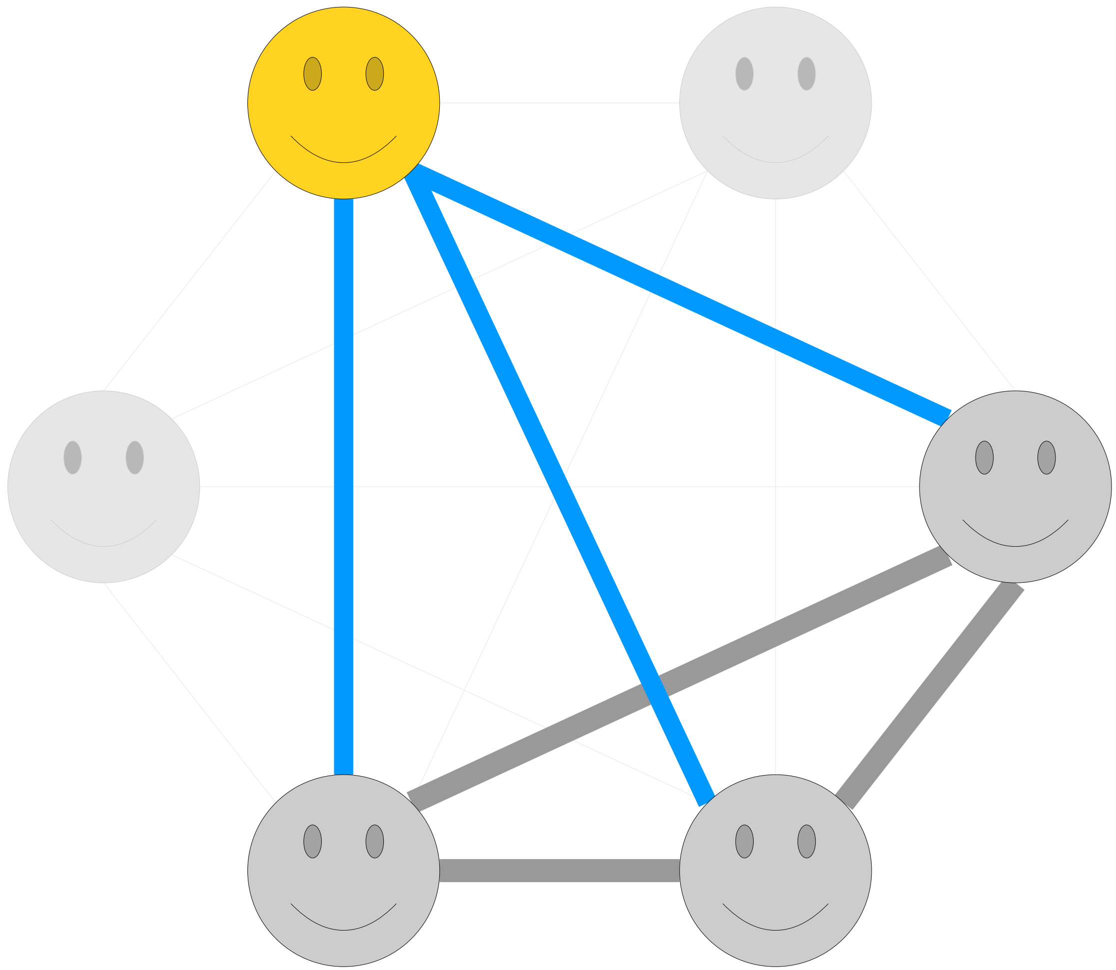


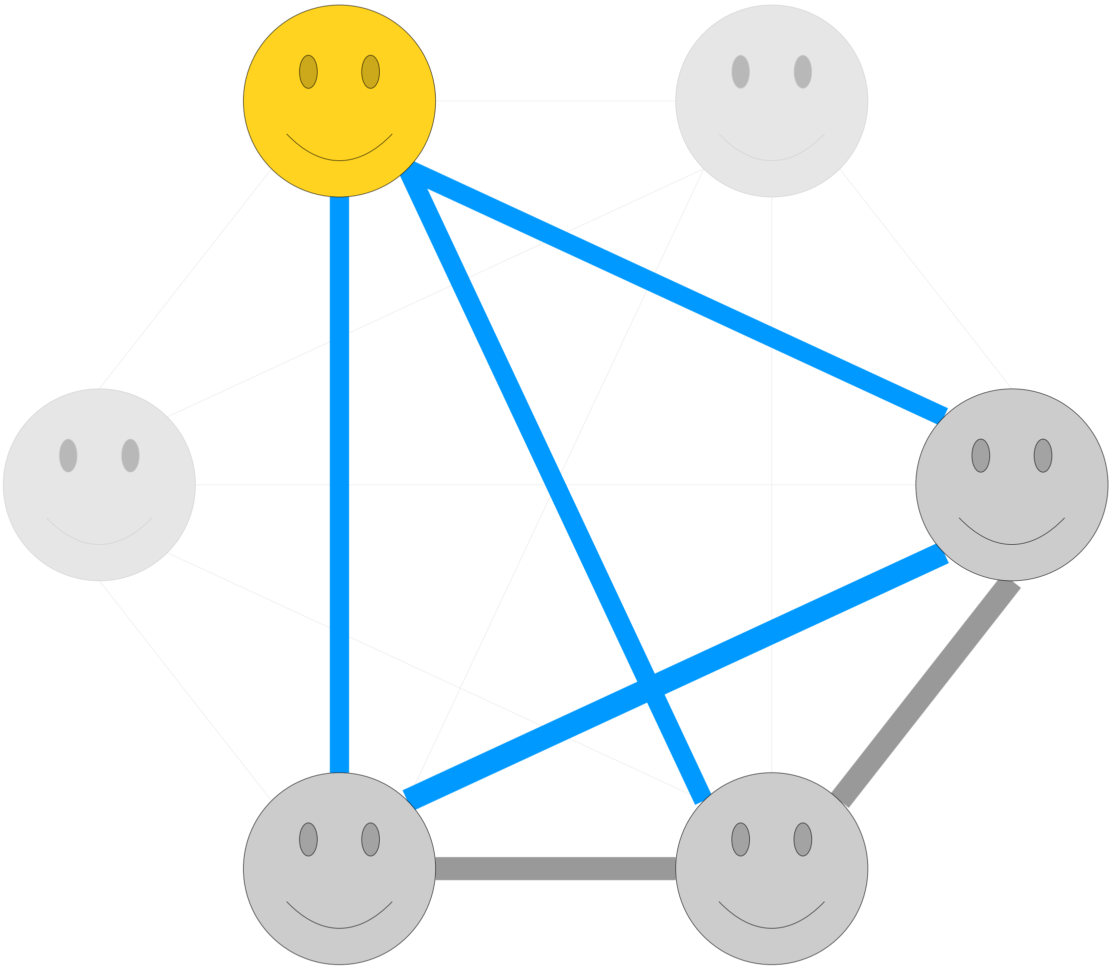


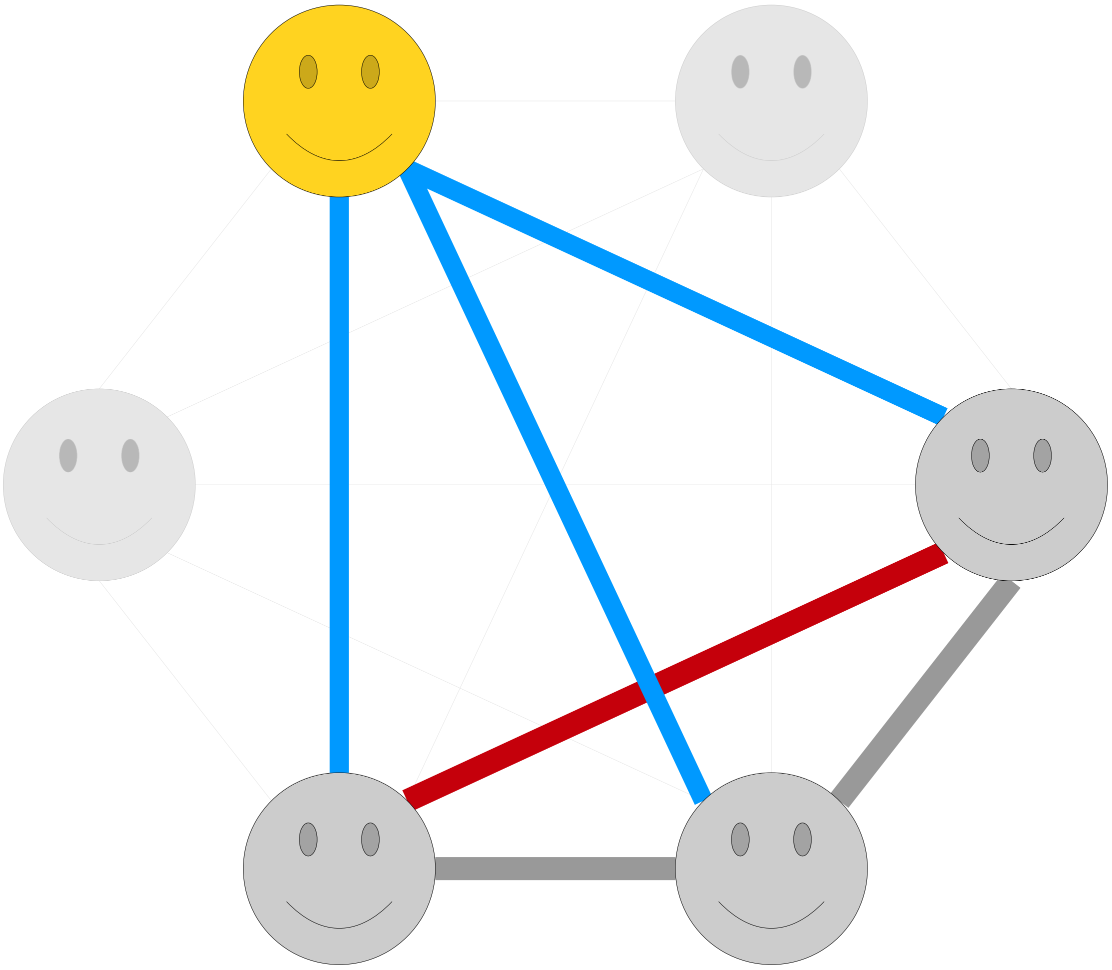


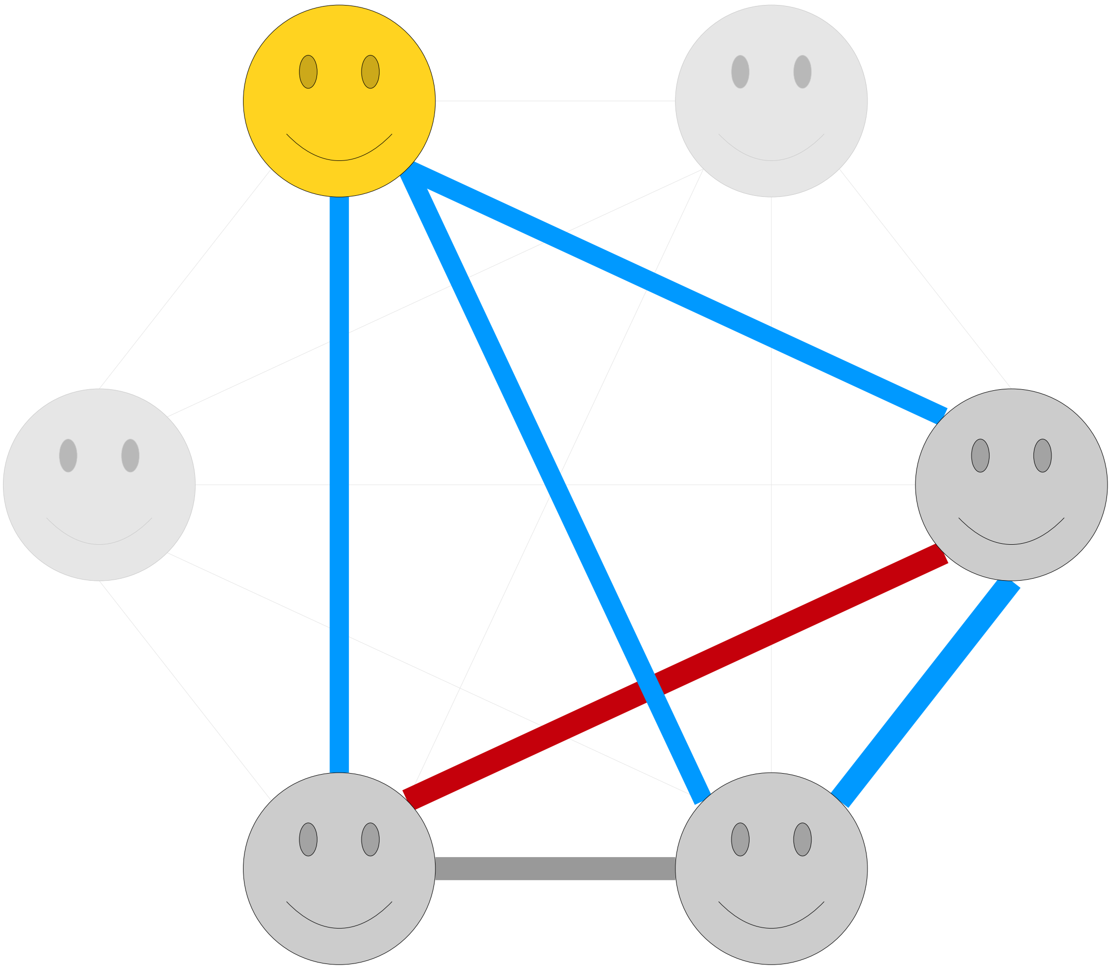
Observation 1: If we pick any node in the graph, that node will have at least $\lceil 5/2 \rceil = 3$ edges of the same color incident to it.

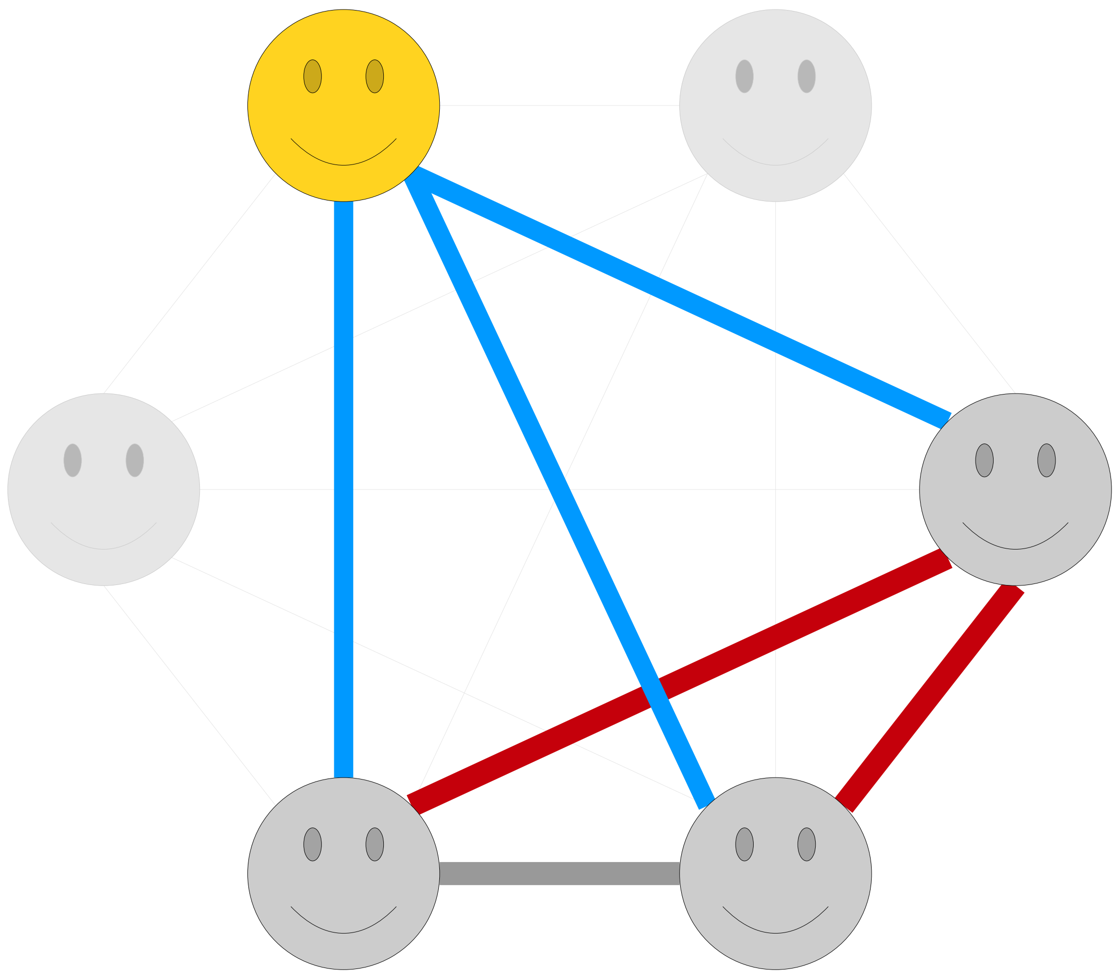


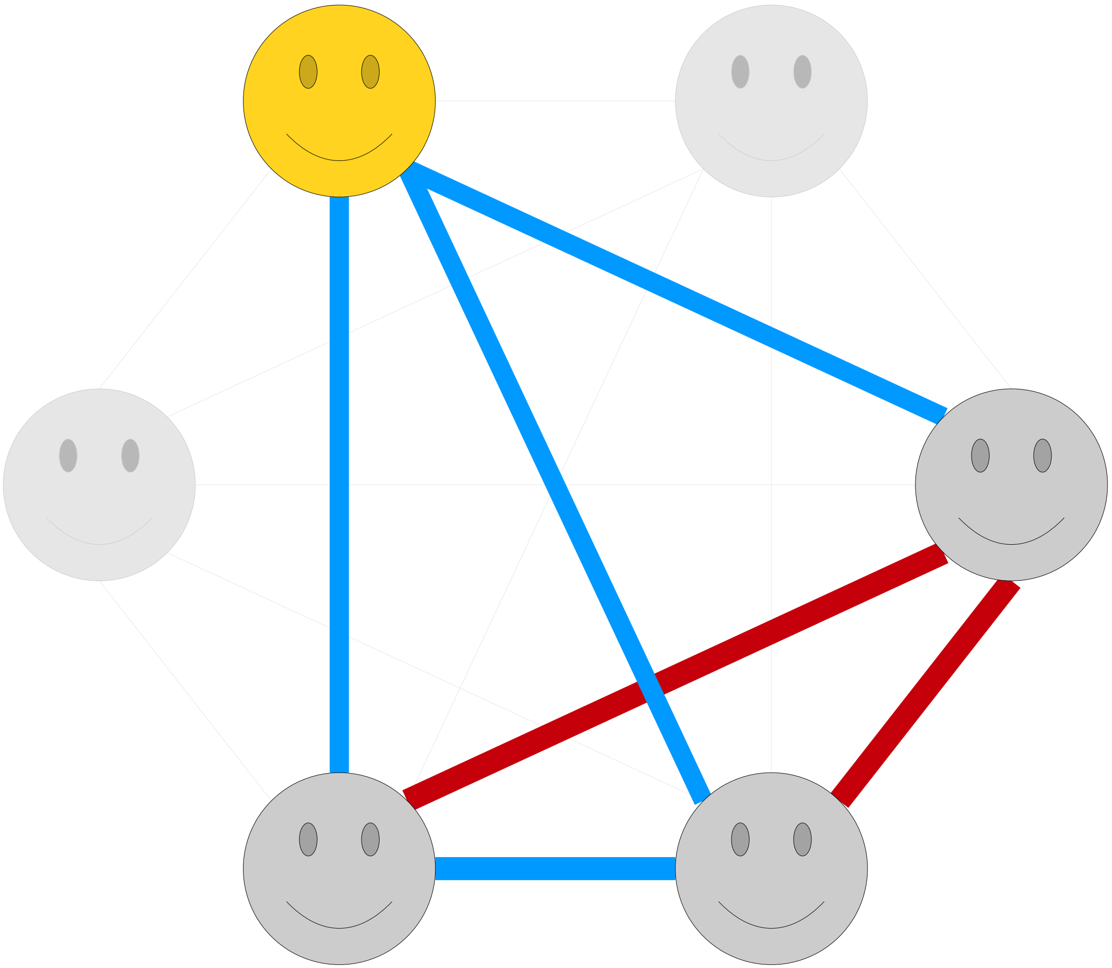


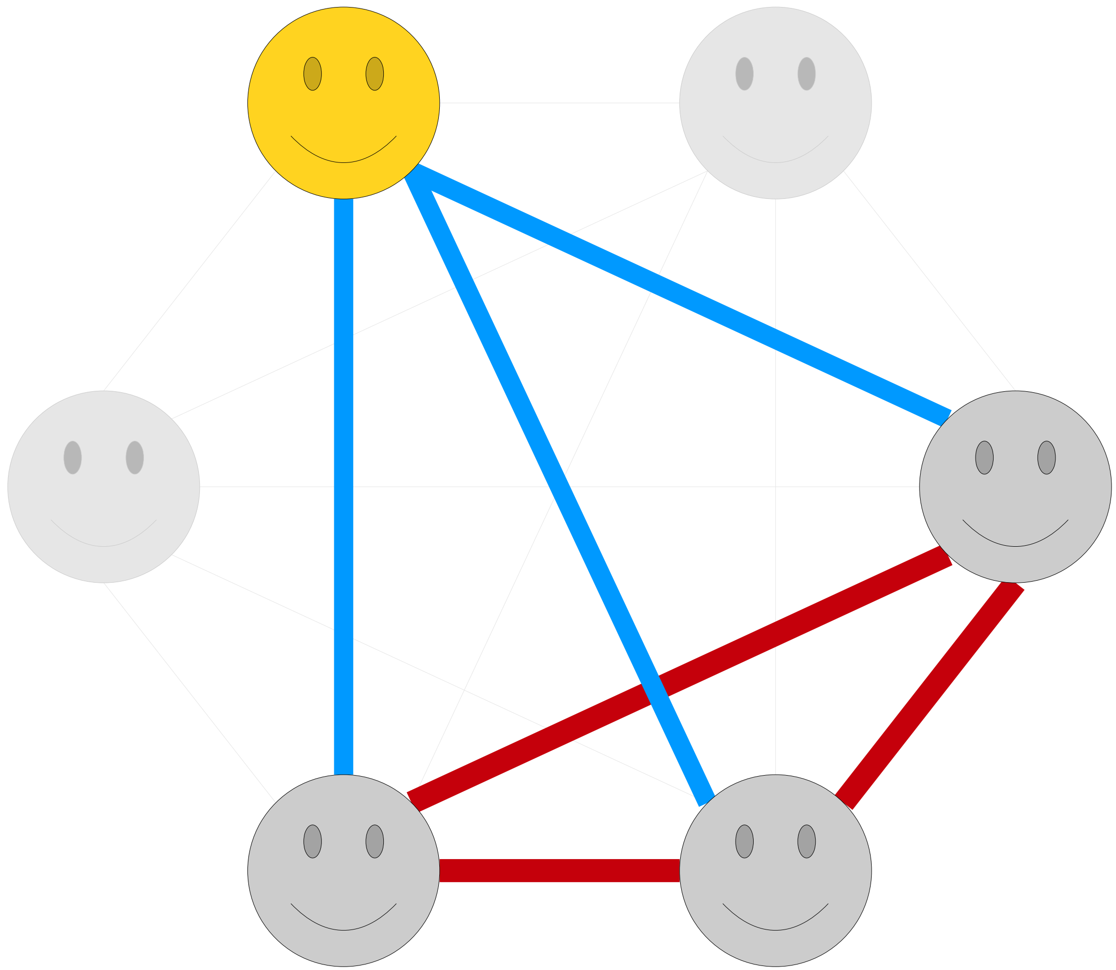












Theorem: Consider a 6-clique in which every edge is colored either red or blue. Then there must be a triangle of red edges, a triangle of blue edges, or both.

Proof: We need to show that the colored 6-clique contains a red triangle or a blue triangle.

Let x be any node in the 6-clique. It is incident to five edges and there are two possible colors for those edges. Therefore, by the generalized pigeonhole principle, at least $\lceil 5/2 \rceil = 3$ of those edges must be the same color. Without loss of generality, assume those edges are blue.

Let r, s, and t be three of the nodes adjacent to node x along a blue edge. If any of the edges $\{r, s\}$, $\{r, t\}$, or $\{s, t\}$ are blue, then one of those edges plus the two edges connecting back to node x form a blue triangle. Otherwise, all three of those edges are red, and they form a red triangle. Overall, this gives a red triangle or a blue triangle, as required. ■

Ramsey Theory

- The theorem we just proved is a special case of a broader result.
- ***Theorem (Ramsey's Theorem):*** For any natural number n, there is a smallest natural number $R(n)$ such that if the edges of an $R(n)$-clique are colored red or blue, the resulting graph will contain either a red n-clique or a blue n-clique.
  - Our proof was that $R(3) \leq 6$.
- A more philosophical take on this theorem: true disorder is impossible at a large scale, since no matter how you organize things, you're guaranteed to find some interesting substructure.

Let's take a quick break!

Time-Out for Announcements!

Problem Set

- Problem Set 2 solutions are up on the course website – please take a look at them as soon as possible.
- TAs are working hard on grading your assignments. We’re aiming to have those returned to you by Wednesday before class.

Back to CS103!

A Little Math Puzzle

"In a group of $n > 0$ people …

- 90% of those people enjoyed *Get Out*,
- 80% of those people enjoyed *Lady Bird*,
- 70% of those people enjoyed *Arrival*, and
- 60% of those people enjoyed *Zootopia*.

No one enjoyed all four movies. How many people enjoyed at least one of *Get Out* and *Arrival*?"

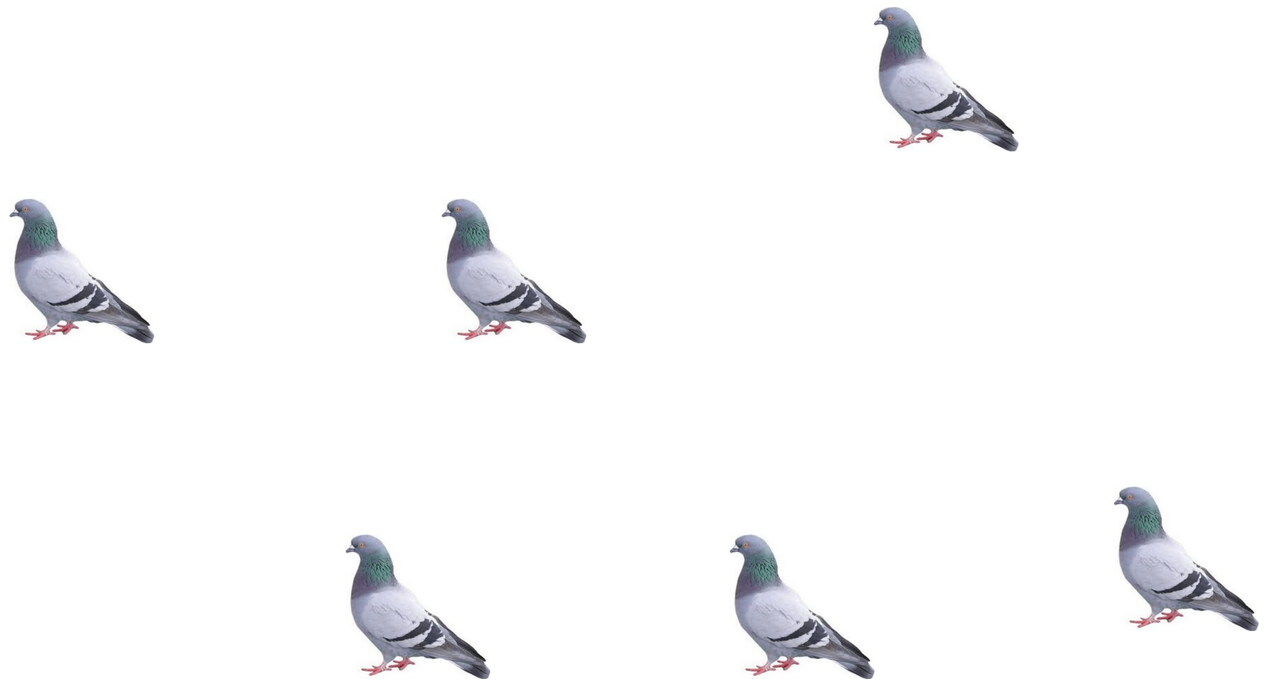
(Adapted from here.)

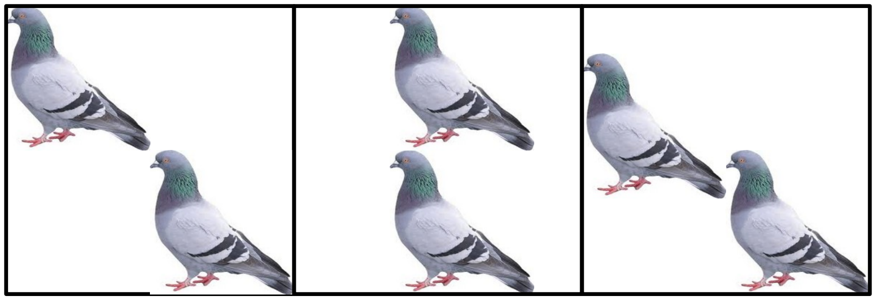
Other Pigeonhole-Type Results

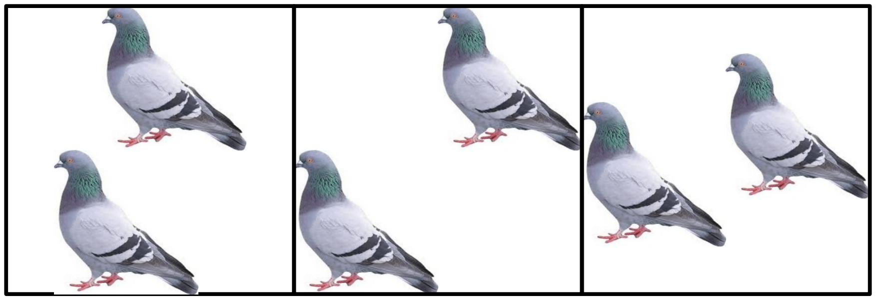
If m objects are distributed into n boxes, then [condition] holds.

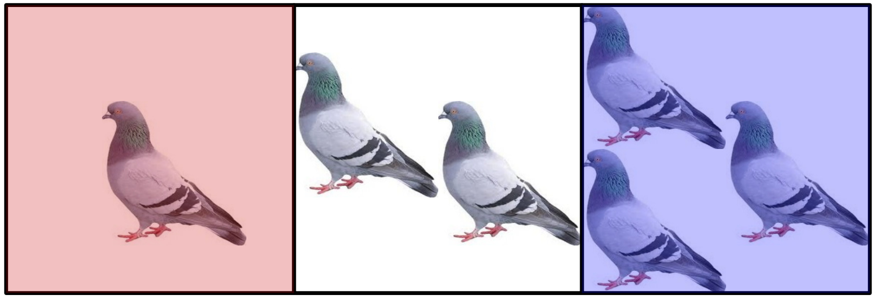
If m objects are distributed into n boxes, then some box is loaded to at least the average $^m/_n$, and some box is loaded to at most the average $^m/_n$.

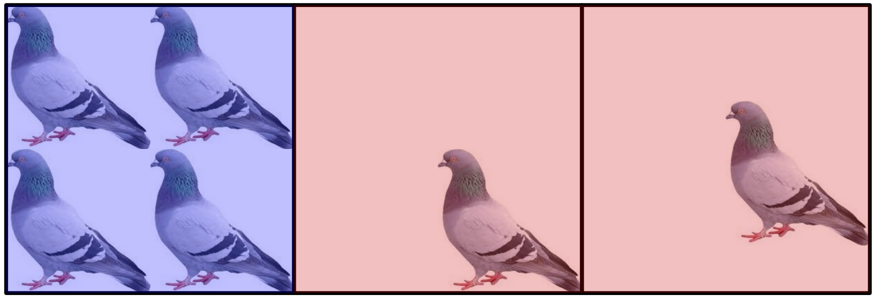
If m objects are distributed into n boxes, then [condition] holds.











Theorem: If m objects are distributed into n bins, then there is a bin containing more than m/n objects if and only if there is a bin containing fewer than m/n objects.

Lemma: If m objects are distributed into n bins and there are no bins containing more than m/n objects, then there are no bins containing fewer than m/n objects.

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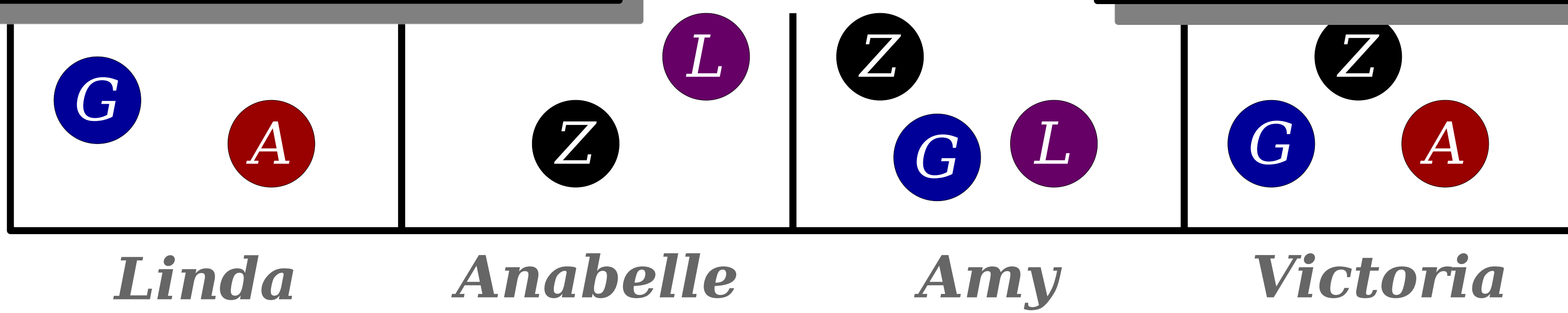
"In a group of $n > 0$ people …

- 90% of those people enjoyed ***Get Out***,
- 80% of those people enjoyed ***Lady Bird***,
- 70% of those people enjoyed ***Arrival***, and
- 60% of those people enjoyed ***Zootopia***.

No one enjoyed all four movies. How many people enjoyed at least one of *Get Out* and *Arrival*?"

Insight 1: Model movie preferences as balls (movies) in bins (people).

Insight 2: There are n total bins, one for each person.



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$$.9n + .8n + .7n + .6n$$
$$= 3n$$

Insight 3: There are $3n$ balls being distributed into n bins.

Insight 4: The average number of balls in each bin is 3.

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Insight 5: No one enjoyed more than three movies…

Insight 6: … so no one enjoyed fewer than three movies …

Insight 7: … so everyone enjoyed exactly three movies.

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Insight 8: You have to enjoy at least one of these movies to enjoy three of the four movies.

Conclusion: **Everyone** liked at least one of these two movies!

Theorem: In the scenario described here, all n people enjoyed at least one of *Get Out* and *Arrival*.

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Proof: Suppose there is a group of n people meeting these criteria.

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and since there are n people, there are n bins.

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Now suppose for the sake of contradiction that someone didn't enjoy *Get Out* and didn't enjoy *Arrival*.

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Going Further

- The pigeonhole principle can be used to prove a *ton* of amazing theorems. Here's a sampler:
  - There is always a way to fairly split rent among multiple people, even if different people want different rooms. *(Sperner's lemma)*
  - You and a friend can climb any mountain from two different starting points so that the two of you maintain the same altitude at each point in time. *(Mountain-climbing theorem)*
  - If you model coffee in a cup as a collection of infinitely many points and then stir the coffee, some point is always where it initially started. *(Brower's fixed-point theorem)*
  - A complex process that doesn't parallelize well must contain a large serial subprocess. *(Mirksy's theorem)*
  - Any positive integer *n* has a nonzero multiple that can be written purely using the digits 1 and 0. *(Doesn't have a name, but still cool!)*

More to Explore

- Interested in more about graphs and the pigeonhole principle? Check out…
  - … ***Math 107*** (Graph Theory), a deep dive into graph theory.
  - … ***Math 108*** (Combinatorics), which explores a bunch of results pertaining to graphs and counting things.
  - … ***CS161*** (Algorithms), which explores algorithms for computing important properties of graphs.
  - … ***CS224W*** (Deep Learning on Graphs), which uses a mix of mathematical and statistical techniques to explore graphs.
- Happy to chat about this in person if you'd like.

Next Time

- ***Mathematical Induction***
  - Reasoning about stepwise processes!
- ***Applications of Induction***
  - To numbers!
  - To anticounterfeiting!
  - To puzzles!